To solve $z^2 + 1 = 0$ we "invent" the number i with the defining property $i^2 = -1$. We then "define" the set of complex numbers as $\mathbf{C} = \{x+iy:x,y\in\mathbf{R}\}$. The notation $z = x+iy$ is known as the **Euler form** of z, **C** is more rigorously defined using the concept of **ordered pair** as $\mathbf{C} = \{(x,y):x,y\in\mathbf{R}\}$. $z = (x,y)$ is the **Hamilton form** of z so that **C** can be identified with $\mathbf{R}^2$. If $z_1 = x_1+iy$, and $z_2 = x_2+iy_2$, then $z_1+z_2 =_{df} (x_1+x_2) + i(y_1+y_2)$ and $z_1z_2 =_{df} (x_1x_2-y_1y_2) + i(x_1y_2+x_2y_2)$. Using these definitions, the nine properties of addition and multiplication in the definition of an **abstract algebraic field** can be proved so that the system (**C**, +, ·,0,1) is an example of an abstract algebraic field. Computation of the product of two complex numbers is made easy using the algebera of **R**, FOIL and the defining property $i^2 = -1$: $(x_1+iy_1)(x_2+iy_2) = x_1x_2+x_1iy_2 + iy_1y_2 + i^2y_1y_2 = x_1x_2 + i\,(x_1y_2 + y_1y_2) - y_1y_2 = (x_1x_2-y_1y_2) + i(x_1y_2+x_2y_1)$. This makes evaluation of polynomial functions easy.

EXAMPLE #1. If $f(z) = (3 + 2i) + (2 + i)z + z^2$, then

$$f(1+i) = (3 + 2i) + (2 + i)(1 + i) + (1 + i)^2 = (3 + 2i) + (2 + 3i + i^2) + (1 + 2i + i^2)$$
$$= (3 + 2i) + (2 + 3i - 1) + (1 + 2i - 1) = (3 + 2i) + (1 + 3i) + (\ 2i\ ) = 4 + 7i.$$

Division and evaluation of rational functions is made easier by using the complex conjugate. The magnitude or absolute value of a complex number is defined as the distance to the origin in the complex plane.

DEFINITION #1. If $z=x+iy$, then the **complex conjugate** of z is given by $\bar{z} = x - iy$. Also the **magnitude** or **absolute value** of z is $|z| = \sqrt{x^2 + y^2}$.

THEOREM #1. If $z,z_1,z_2\in\mathbf{C}$, then a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, b) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$, c) $|z|^2 = z\,\bar{z}$, d) $\bar{\bar{z}} = z$.

In addition to the **rectangular representation** given above, complex numberscan be represented using **polar coordinates**: $z = x + iy = r\cos\theta + i\sin\theta = r(\cos\theta + i\sin\theta) = r\,\underline{/\ \theta}$ (polar). Note that $r = |z|$. For example, $2\,\underline{/\pi/4} = \sqrt{2} + i\sqrt{2}$ and $1 + \sqrt{3}\ i = 2\ \underline{/\pi/3}$. If $z_1 = 3 + i$ and $z_2 = 1 + 2i$, then

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2} = \frac{(3+i)(1+2i)}{(1+2^2)} = \frac{(3+6i+i+2i^2)}{(1+4)} = \frac{(3-2+7i)}{5} = \frac{1}{5} + \frac{7}{5}i$$

EXAMPLE # 2 If $f(z) = \dfrac{z+(3+i)}{z+(2+i)}$, then

$$f(4+i) = \frac{(4+i)+(3+i)}{(4+i)+(2+i)} = \frac{(7+2i)}{(6+3i)} = \frac{(7+2i)}{(6+3i)}\ \frac{(6-3i)}{(6-3i)} = \frac{(42+6)+(12-21)i}{(36+9)} = \frac{48-9i}{45}$$
$$= \frac{48}{45} - \frac{9}{45}i = \frac{16}{15} - \frac{1}{5}i$$

THEOREM #2. If $z_1 = r_1 \angle\theta_1$ and $z_2 = r_2 \angle\theta_2$. Then a) $z_1 z_2 = r_1 r_2 \angle\theta_1 + \theta_2$, b) If $z_2 \neq 0$, then $\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle\theta_1 - \theta 2$, c) $z_1^2 = r_1^2 \angle 2\theta_1$, d) $z_1^n = r_1^n \angle n\theta_1$.

EULER'S FORMULA. By definition $e^{i\theta} =_{df} \cos\theta + i \sin\theta$. This gives another way to write complex numbers in polar form:

$z = 1 + i\sqrt{3} = 2 \angle \pi/3 = 2e^{i\pi/3}$ and $z = \sqrt{2} + i\sqrt{2} = 2 \angle \pi/4 = e^{i\pi/4}$

More importantly, it can be shown that this definition allows the extension of exponential, logarithmic, and trigonometric functions to complex numbers and that the standard properties for these functions still hold. **This is nothing short of amazing!!** It allows you to determine what these extensions should be and to evaluate these functions for all complex numbers. Hence it allows you to determine appropriate extensions to **C** for all elementary function.

EXAMPLE #3. If $f(z) = (2 + i)\, e^{(1+i)z}$, find $f(1 + i)$.

Solution. First $(1 + i)(1 + i) = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$. Hence

$f(1 + i) = (2 + i)\, e^{2i} = (2 + i)(\cos 2 + i \sin 2) = 2 \cos 2 + i(\cos 2 + 2 \sin 2) + i^2 \sin 2$

$= 2 \cos 2 - \sin 2 + i(\cos 2 + 2 \sin 2)$ (exact answer)

$\approx -1.7415911 + i(1.4024480)$ (approximate answer)

How do you know that $-1.7415911 + i(1.4024480)$ is a good approximation to $2 \cos 2 - \sin 2 + i(\cos 2 + 2 \sin 2)$? Can you give an expression for the distance between these two complex numbers?

If Euler's formula holds for all complex numbers $z = \theta \in \mathbf{C}$, then

$$e^{iz} = \cos z + i \sin z. \qquad (1)$$

If sin z and cos z have the properties $\cos(-z) = \cos(z)$ and $\sin(-z) = -\sin(z)$, then

$$e^{-iz} = \cos(-z) + i \sin(-z) = \cos(z) - i \sin(z). \qquad (2)$$

Adding (1) and (2) we obtain $2 \cos(z) = e^{iz} + e^{-iz}$. Hence if $z = x + iy$, then we should have

$$\begin{aligned}\cos(z) &= (1/2)(e^{iz} + e^{-iz}) = (1/2)(e^{i(x+iy)} + e^{-i(x+iy)}) = (1/2)(e^{-y+ix} + e^{y-ix}) \\ &= (1/2)[e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x)] \\ &= [\cos(x)][(1/2)(e^{y} + e^{-y})] + i[\sin(x)][(1/2)(e^{y} - e^{-y})] \qquad (3)\end{aligned}$$

which gives the appropriate definition of $\cos(z) = \cos(x+iy)$.

EXAMPLE #4. If $f(z) = z \cos[(1 + i) z]$, find $f(1 + i)$.

Solution. First $(1 + i)(1 + i) = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$. Hence

$f(1 + i) = (1 + i) \cos(2i) = (1 + i)\{[\cos(0)][(1/2)(e^2 + e^{-2})] + i[\sin(0)][(1/2)(e^2 - e^{-2})]\}$

$= (1 + i)[(1/2)(e^2 + e^{-2})] = \cosh(2) + i \cosh(2) = [(e^2 + e^{-2})/2] + i[(e^2 + e^{-2})/2]$

$= [(e^4 + 1)/(2e^2)] + i\,[(e^4 + 1)/(2e^2)]$ (exact answer)

$\approx 3.694528049 + i\,3.694528049$ (approximate answer)

How do you know that $3.694528049 + i\,3.694528049$ is a good approximation to $[(e^4 + 1)/(2e^2)] + i\,[(e^4 + 1)/(2e^2)]$? Can you give an expression for the distance between these two complex numbers? Try sketching them in the complex plane.

The **truth value** (i.e. either **true** or **false**) of an **open statement** (e.g. an **equation** such as $x^2 + 1 = 0$) depends on the value of the "**variable(s)**" in the statement. To be useful, this assumes that there is some method of deciding whether a given statement is true. The process may be difficult and even unknown but it is assumed to exist. Finding the values which make the statement true solves a "**problem**". Even if there is an easy process to determine if a given element is a solution, there may be no process to find all such elements. We call the set of all possible values of the variable, the **domain** for the open statement. Although this is technically a different use of the word than in connections with functions, it is certainly in agreement with the general use of the word. And, with the correct interpretation, it is in agreement with its use in connection with functions.

An equation is **satisfied** if substituting a value from its domain D yields a true statement. That is, if the left hand side (LHS) of the equation is the same as the right hand side (RHS). Unless these are trivially **identical**, then some **operations** must be performed to compute both sides. We may think of these as functions from D to some other set, say CoD and our equation becomes a mapping problem: $f(x) = g(x)$. That is, we wish to find those $x \in D$ for which f and g have the same values i.e., map to the same value in CoD. If D is a set of **numbers**, these functions might be defined by combinations of **binary operations** using the algebra of the number system. If CoD also has an algebra (e.g., if CoD = D or if they are both part of the same number system) we may then develop an **algebra of functions**. If we can add in CoD, CoD has an additive identity element (called 0), and each element $y \in$CoD has an additive inverse (called $-y$, we may define $h(x) = f(x) - g(x)$ so that our equation becomes $h(x) = 0$. Our problem now is a **mapping problem** in the sense that for the function h we wish to determine all of the elements in D that map into the element 0. The set $\{x \in D: h(x) = 0\}$ is called the **null set** of h. Even without any structure, if g(x) is a constant function, say $g(x) = b$, then our problem $f(x) = b$ is the mapping problem: Find all those elements in D that map into b (i.e., find $f^{-1}(\{b\})$).

It is useful to view problems as mapping problems when properties of the functions f, g, and h are known. Suppose f provides a one-to-one correspondence between D and CoD. Then f has an inverse function f^{-1} and the unique solution of the problem $f(x) = b$, is $x = f^{-1}(b)$. Ones knowledge of f , perhaps provided by the algebraic structures on D and CoD, will determine ones ability to actually obtain x for any given b.

EXAMPLE: Suppose $f:\mathbf{R}\to\mathbf{R}$, that $f(2) = 3$, $f^{-1}(\{3\}) = \{2\}$, and that f is odd. Solve $f(x) = -3$.
Solution: Since $f(2) = 3$ and f is odd we have that $f(-2) = -f(2) = -3$ so that -2 is a solution. This proves existence of at least one solution. To obtain uniqueness (i.e., that this is the only solution) note that since $f^{-1}(\{3\}) = \{2\}$, we have that 2 is the only element that gets mapped into 3. Hence -2 is the only element that gets mapped into -3. To see this, suppose not. Let $x\neq-2$ be some other real number such that $f(x) = -3$. Then, since f is odd, $f(-x) = -f(x) = -(-3) = 3$. But 2 is the only element that is mapped into 3. Hence $-x = 2$. Hence $x = -2$. This **contradicts** our assumption that $x\neq-2$. Hence that assumption must be false so that -2 is indeed the only element that gets mapped into -3 and is the only solution to the problem. Our **solution set** is therefore $S = \{2\}$. In order to solve the problem, we did not need to be able to find f(x) for all

real numbers x. Our limited knowledge of f was sufficient to solve the problem.

We can prove both uniqueness (the forward process) and existence (the reverse process) together as follows since each step in the **solution process** results in an equivalent problem.

STATEMENT	REASON
$f(x) = -3$	Statement of Problem.
$-f(x) = 3$	Algebraic property for obtaining equivalent equations in **R** (i.e, equations with the same solution set). In this case it is "multiplying both sides of the equation by -1".
$f(-x) = 3$	Assumption that f is odd.($f(-x) = -f(x)$.)
$-x \in f^{-1}(\{3\})$	Definition of f^{-1} of a set. ($f^{-1}(B) = \{x \in D_f : f(x) \in B\}$; that is, $f^{-1}(B)$ is the set Of all elements in the domain of f that get mapped into the set B.
$-x \in \{2\}$	$f^{-1}(\{3\}) = \{2\}$ is given.
$-x = 2$	Definition of an element of a set. (If $-x \in \{2\}$ and 2 is the only element in the set, then $-x = 2$.)
$x = -2$	Algebraic property for obtaining equivalent equations in **R**. (Multiplying both sides of the equation by -1).

Let us be more specific. Let $f: D \to \mathbf{K}$, **K** be a field, $D \subseteq \mathbf{K}$, and consider the scalar equation $f(x) = b$ as a mapping problem. That is, rather than view the equation as being solvable by using the equivalent equation operations (EEO's) that follow from the field operations of adding (subtraction) and multiplication (division), we wish to determine what properties of the function f will assure us that there is a (i.e., at least one) solution (the existence problem) and what properties assure us that there is at most one solution (the uniqueness problem).

<u>DEFINITION #1</u>. Let $f: D \to \mathbf{K}$. Then the **range** of f is the set R_f (or $R(f)$) $= \{y \in \mathbf{K}: \exists\, x \in D \text{ s.t. } f(x) = y\}$.

<u>THEOREM #1</u>. If $b \in R_f$ = the range of f, then $f(x) = b$ has at least one solution.

Proof. Suppose $b \in R_f$. This just means that at least one $x \in D$ maps to b. Hence this x is a solution of $f(x) = b$. Q.E.D.

Note that we have not really solved $f(x) = b$ since we do not know what x maps to b, but have simply assumed that $b \in R(f)$ so that we know that (at least) one such x exists.

<u>DEFINITION #2</u>. Let $f: D \to \mathbf{K}$. The function f is said to be **onto** (**surjective**) if $R_f = \mathbf{K}$. (That is, f is onto if the range is the entire co-domain.)

THEOREM #2. If $R_f = \mathbf{K}$, then $f(x) = b$ has at least one solution for all $b\in\mathbf{K}$.

Proof. Suppose $\mathbf{K} = R_f$. Then no matter what $b\in\mathbf{K}$ is chosen, we have $b\in R_f$. Hence at least one $x\in D$ maps to b. Hence this x is a solution of $f(x) = b$. Hence $f(x) = b$ has at least one solution for all $b\in\mathbf{K}$ Q.E.D.

If $R_f = \mathbf{K}$, we say that f satisfies the existence property for $f(x) = b$ (as $f(x) = b$ has a solution no matter how $b\in\mathbf{K}$ is chosen.

DEFINITION #3. Let $f: D \to \mathbf{K}$. The function f is said to be **one-to-one** (**injective**) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. (This is the **contra positive** of the statement, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. That is, distinct values of x in X get mapped to distinct values of y in Y. Recall that the negation of a negative is the positive.) .

THEOREM #3. If f is one-to-one, then $f(x) = b$ has at most one solution.

Proof. Suppose f is one-to-one and that x_1 and x_2 are solutions to $f(x) = b$. Then $f(x_1) = b$ and $f(x_2) = b$ so that $f(x_1) = f(x_2)$. Since f is one-to-one, we have by the definition that $x_1 = x_2$. Since x_1 and x_2 must be the same, there is at most one solution to $f(x) = b$.

Q.E.D.

DEFINITION #4. Let $f: D \to \mathbf{K}$. If f is both 1-1 and onto, then it is said to be **bijective** or to form a **one-to-one correspondence** between the domain and the co-domain. (If f is 1-1, then it always forms a 1-1 correspondence between its domain and its range.) The **identity function** (denoted by I or i_X) from $\mathbf{K}$ to itself is the function $I:\mathbf{K} \to \mathbf{K}$ defined by $I(x) = x \ \forall x \in \mathbf{K}$.

THEOREM #4. Let $f: D \to \mathbf{K}$. If f is bijective, $f(x) = b$ always has exactly one solution.

DEFINITION #4. Let $f:D \to \mathbf{K}$. If f is bijective, then its inverse function, $f^{-1}:\mathbf{K}\to\mathbf{K}$ can be defined by
$y = f^{-1}(x)$ if and only if $x = f(y)$. (e.g., $y = \ln x$ if and only if $x = e^y$ where $f(x) = e^x$, $f:\mathbf{R}\to[0,\infty)$ and $f^{-1}(x) = \ln x$, $f^{-1}:[0,\infty)\to\mathbf{R}$.

THEOREM #5. Let $f:D \to \mathbf{K}$. If f is bijective with inverse function, $f^{-1}:\mathbf{K}\to\mathbf{K}$, then the unique solution of $f(x) = b$ is $x = f^{-1}(b)$.

If f is bijective with inverse function, $f^{-1}:\mathbf{K}\to D$, when we say that the unique solution of $f(x) = b$ is $x = f^{-1}(b)$ we have not really solved anything, but have just transferred the problem of finding the solution to that of finding the inverse function. This may be possible, but is often more difficult then just solving the problem. Even so, the concept of an inverse function is very useful in thinking about how many solutions $f(x) = b$ might have.

We now consider the relationship between the problems defined by the equation $f\circ g(x) = 0$ and $f(y) = 0$ where $f:D\to\mathbf{R}$ and $g: \mathbf{R}\to \mathbf{R}$ is a bijection. Thus we compare the null set of f with that of $f\circ g$.

THEOREM #1. Let $f:D\to\mathbf{R}$ be a function and $g: \mathbf{R}\to \mathbf{R}$ be a bijecton. Then $f(y) = 0$ if and only if $f\circ g(x) = 0$ where $y = g(x)$.

Proof. Suppose $f(y) = 0$ and $y = g(x)$. Then $f\circ g(x) = f(g(x)) = 0$. On the other hand, suppose $f\circ g(x) = 0$ and $y = g(x)$. Then $f(y) = 0$. Q.E.D.

COROLLARY #2. Let $g: \mathbf{R}\to \mathbf{R}$ be a bijecton. Then $y\in N_f$ if and only if $x\in N_{f\circ g}$.

Thus we may find the zeros of $f\circ g$ by first finding the zeros of f and then computing g^{-1}.

EXAMPLE. Solve $\sin(3x+1) = 0$.
Solution. Let $y = g(x) = 3x+1$. Since the zeros of $\sin(y)$ are $y = n\pi$ with $n\in\mathbf{Z}$ and $g^{-1}(y) = (y - 1)/3$, we see that the zeros of $\sin(3x+1)$ are $(n\pi - 1)/3$. We may write the solution as
$\sin(3x+1) = 0, \Rightarrow 3x+1 = n\pi$ with $n\in\mathbf{Z}$, $\Rightarrow x = (n\pi - 1)/3$ with $n\in\mathbf{Z}$.

What is the relationship between the zeros of f and $f\circ g$ if g is not a bijection? If g is a bijection on **R** and we know the function $h = f\circ g$, what is the function f in terms of h and g?

Let $f:D\to\mathbf{R}$ be a function and $g: \mathbf{R}\to \mathbf{R}$ be a bijecton. Then the problem Prob(D, $f\circ g(x) = 0$) and Prob(**R**, $f(x) = 0$) are said to be simultaneously solvable. That is, we know what elements are i the null space of $f\circ g$ exactly when we know what is in the null space of f. We say that $y = g(x)$ is a change of variables. For example, letting $y = 3x + 1$ is a change of variables that converts
$\sin(3x+1) = 0$ into the simultaneously solvable problem $\sin(y) = 0$ since $y =3x+1$ is a bijection from **R** to **R**.