A SERIES OF CLASS NOTES TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

REMEDIAL CLASS NOTES

A COLLECTION OF HANDOUTS FOR REMEDIATION

IN FUNDAMENTAL CONCEPTS FOR

PROBLEM SOLVING IN MATHEMATICS

CHAPTER 2
Problem Solving in Mathematics with Emphasis on Scalar Equations and Inequalities

Handout #1 PROBLEM SOLVING IN MATHEMATICS Professor Moseley
Chap. 2

According to Paul Halmos the heart of mathematics is its **problems**. A problem, according to Webster, is a question raised for inquiry, consideration, discussion, decision, or solution. Webster indicates that **a problem in mathematics** or physics requires something to be done. The thing to be done answers the question and is referred to as the **solution**. But **formulation and solution of problems in mathematics** may mean different things to different people depending on their interest and on their level of mathematical development. We could attempt to develop a general framework for the formulation of all types of problems in mathematics, but we would most certainly fail. On the other hand, most mathematical problems seem to fall into three categories:

1.Problems with an established algorithm for solution (e.g. computational problems). Such problems will be referred to as **evaluation problems**. The solution process for such a problem answers the question "How to find?" or "How to compute?' Students can be trained to carry out these **mathematical procedures** or **computational skills**.

2. Problems defined by equations, inequalities or other properties. Such problems will be referred to as **find or locate problems**. They ask the question "If any, which ones?" There may or may not be a "How to find" algorithm associated with the problem. If there is, it can be applied. If not, the problem becomes developing such an algorithm. If the algorithm requires an infinite number of steps, we need the concept of an **approximate solution**.

3. Theory problems. Such problems will be referred to as **think problems**. They ask the question "Why?" Why does a particular algorithm work for one problem, but not for a similar problem? What is the set of "find" problems that a particular algorithm does work for and why? How can we develop solution procedures for all problems of interest. These results are often given in the development of a mathematical theory using a definition/theorem/proof format.

Learning to **solve evaluation problems** means being trained (or training oneself) to use known processes or algorithms on specific examples. At the other extreme in problem solving is the **development of a mathematical theory** which may then lead to the development of algorithms for solving "find" problems. Theory development requires an understanding of what is already known (i.e. what has been proved) and hence an ability with proofs. We focus on problems (of the type useful to engineers, scientist, and applied mathematicians) between these two extremes by examining a framework which generalizes the problem of solving scalar equations; that is, we consider **find or locate problems**. This framework assumes in the problem formulation that you understand what is meant by solving an evaluation problem but not that you can write proofs or develop a mathematical theory.

A FRAMEWORK FOR FORMULATING "FIND" PROBLEMS. In general in mathematics, a set with structure is called a **space**. We say that a ("find") problem, call it Prob, is well-formulated in a set theoretic sense if:

1. There is a clearly defined space, call it Σ, where, if there are any, we wish to find all **solutions** to the problem.
2. There is a clearly defined **property** or **condition**, call it C, that the solution elements in Σ and only the solution elements satisfy. This property is defined by the structure on Σ.

The need for clearly specifying the set Σ is illustrated by the equation $x^2 + 1 = 0$. The existence of a solution depends on whether we choose the real numbers **R** or the complex numbers **C** as the set which must contain the solution. Both are examples of the abstract algebraic structure known as a **field** where addition (subtraction) and multiplication (division except by zero) give the algebraic structure.

Webster's definition causes some confusion as to what is meant by **the solution** of a problem. For an evaluation problem, the **solution process or algorithm used** is sometimes referred to as **the solution**. **The answer obtained** is sometimes referred to as **the solution**. In our framework for "find" problems (FFP's), **a solution** is an element in Σ that satisfies the property C(x) and **the solution set** is the set containing all solutions, $S = \{x \in \Sigma: C(x)\}$. Since Σ and C define the problem Prob, we let $\text{Prob}(\Sigma,C) = \{x \in \Sigma: C(x)\}$ and think of $\text{Prob}(\Sigma,C)$ as an **implicit description** of the **solution set**. **The solution process** is whatever algorithm is used to obtain an **explicit description** of S. We use $\text{Soln}(\Sigma,C)$ to mean the explicit description of the solution set obtained by the solution process. Since as sets we have $\text{Prob}(\Sigma,C) = \text{Soln}(\Sigma,C)$, for brevity in working examples, we let $S = \{x \in \Sigma: C(x)\} \subseteq \Sigma$ be the solution set during the solution process.

If the validity of C(x) is easily checked for any element in Σ, we say that solutions to the problem are **testable**. Solutions of equations are usually testable. Optimization problems define a property that is usually not easily testable. Normally Σ is large or infinite (e.g. **R** and **C**) so that it is not possible to solve the problem by testing each element in Σ individually. Problems where Σ is small enough so that a check of its elements by hand is possible are usually considered to be trivial. On the other hand, some problems where Σ is large but not to large (e.g. Which students at a university have brown eyes?) yield to the technique of testing each element in Σ by using computers and data bases.

SCALAR EQUATIONS. A major class of problems included in this framework are those defined by scalar equations (e.g. $x^2 + 2x = 3$) for a number system. If the number system is a **field** so that additive inverses always exist, by moving all terms to the same side of the equation (i.e., adding appropriate additive inverses to both sides of the equation), we may reformulate these as $f(x) = 0$ where f is a **function** (e.g. $x^2 + 2x - 3 = 0$ so that $f(x) = x^2 + 2x - 3$). The set Σ is the **domain** of f and is a subset of the number system. The condition $f(x) = 0$ is satisfied if the equation is satisfied and is tested by the evaluation of the function f. Hence the solution set, $S = \{x \in \Sigma: f(x) = 0\}$, is the set of **zeros of the function** f. Σ is the domain of the function f and the solution set becomes the **nullset** of f; that is, the set of numbers that are mapped into zero by f. Often, the set Σ where we are to look for solutions (e.g. the domain of the function f) is not specified explicitly when the problem is stated. It may be implicitly understood or it may be left up to the problem solver to decide on an appropriate choice (e.g. **R** or **C**). In any case, determining the set Σ (starting with a knowledge of the elements we allow to be solutions) is essential to a good mathematical formulation of a "find" problem.

INEQUALITIES. Besides equalities, our framework includes inequalities (e.g. $|x - 3| < 4$) which (in an **ordered field**) may be reformulated as $f(x) < 0$ (e.g. $|x - 3| - 4 < 0$ so that $f(x) = |x - 3| - 4$), and the "less than" symbol < may instead be any of $\leq$, $>$, or $\geq$. The set Σ is the domain of f (e.g., it may be a subset of **Q** or **R** as, unlike the rational and the real numbers, the complex numbers do not have a natural ordering). Testing of possible solutions is effected by evaluation of the function f. The solution set is $S = \{x \in \Sigma: f(x) < (\text{alternately}, \leq, >, \text{or} \geq)\ 0\}$ and is the set of numbers x in Σ such that f(x) is less than (alternately, less than or equal to, greater than, or greater than or equal to) zero. Although there might be no solution (e.g. $x^2 < -1$) or one solution (e.g. $x^2 \leq 0$), the solution set when $\Sigma \subseteq \mathbf{R}$ is usually a portion of the real number line and hence is usually an infinite set.

<u>SUMMARY.</u> Although it does not encompass all problem types, the **framework for find problems** (FFP's) discussed above does provide a standard problem solving context for high school students and college undergraduates at the freshman and sophomore level. In addition to scalar equations and inequalities, it can be used for systems of equations, systems of inequalities, difference equations, and differential equations. A clear understanding of this framework should help you toward a better understanding of why problems may have no solution (e.g. $3x-1=(6x+2)/2$), one solution (e.g. $3x-1 = 4x+2$), more than one solution (e.g. $x^2-4 = 0$ and $x^5(x-2)(x-4) = 0$), or even an infinite number of solutions (e.g. $3x+1 = (6x+2)/2$ and the inequality $|x-3|-4 < 0$). This should help you to understand that when we use our technical definition of a solution to a find problem, **not every math problem has exactly one solution**. It should also help you to begin to move from just focusing on being trained (or training yourself) to use algorithms for the solution of evaluation problems to the more advanced view of, given a problem that is well formulated, how does one find answers to the questions: Does a solution exist? Is it unique? How do we know? Can we develop algorithms to find all of the solutions? What other problems will our algorithms solve and why? Hopefully, this will encourage you to spend time trying to understand the "why"s of solving problems in mathematics as well as time training yourself (i.e., doing homework) in the "how to"s illustrated in class.

We can view the real numbers as part of the sequential process of constructing the number systems, starting with **N**, then **Z**, then **Q**, then **R**, and finally **C**. The real numbers **R** can also be viewed geometrically as being in one-to-one correspondence with a line in space. We call this the **real number line** and view **R** as a model for one dimensional space. We also view **R** as being a fundamental model for time. However, we wish to now view **R** as a space or **axiomatic system**; that is, as a set with algebraic and analytic properties defined by axioms. This removes the need to view **R** geometrically or temporally or to provide a lot of details for the construction of **N**, **Z**, and **Q**. We organize the fundamental properties of the real number system into groups. The first three are the standard **axiomatic properties** of **R**:

1) The **algebraic** (**field**) properties. (An **algebraic field** is an abstract algebraic structure. Informally, it is a number system where you can add, subtract, multiply, and divide (except by zero). Examples of an algebraic field are **Q**, **R**, and **C**. However, **N** and **Z** are not fields.)
2) The **order** properties. (These give **R** its one dimensional nature.)
3) The **least upper bound property**. (This leads to the **completeness property** that insures that **R** has no "holes" and may be the hardest property of **R** to understand. **R** and **C** are complete, but **Q** is not complete)

All other properties of the real numbers follow logically from these axiomatic properties. Being able to "see" the geometry of the real number line may help you to intuit other properties of **R**. Although helpful at times, this ability is not necessary for the axiomatic development of the properties of **R**, and can obscure the need for a proof of an "obvious" property. We consider other properties that can be derived from the axiomatic ones. To solve **equations** we consider
4)Additional algebraic (field) properties of **R** including cancellation laws and **factoring** of polynomials.
To solve **inequalities**, we consider:
5) Additional order properties including the definition and properties of the **order relations** $<$, $>$, $\leq$, and $\geq$.

ALGEBRAIC (FIELD) PROPERTIES. Since the set **R** of real numbers along with the **binary operations** of addition and multiplication satisfy the following properties, the system (**R**, +, ·) is an example of a **field** which is an abstract algebraic structure.

A1.	$x+y = y+x \ \ \forall x,y \in \mathbf{R}$	Addition is Commutative
A2.	$(x+y)+z = x+(y+z) \ \ \forall x,y\ z \in \mathbf{R}$	Addition is Associative
A3.	$\exists\ 0 \in \mathbf{R}$ s.t. $x+0 = x \ \ \forall x \in \mathbf{R}$	Existence of a Right Additive Identity
A4.	$\forall x \in \mathbf{R}\ \ \exists\ w$ s.t. $x+w = 0$	Existence of a Right Additive Inverse
A5.	$xy = yx \ \ \ \forall x,y \in \mathbf{R}$	Multiplication is Commutative
A6.	$(xy)z = x(yz) \ \ \ \forall x,y\ z \in \mathbf{R}$	Multiplication is Association
A7.	$\exists\ 1 \in \mathbf{R}$ s.t. $x \cdot 1 = x \ \ \forall x \in \mathbf{R}$	Existence of a Right Multiplicative Identity
A8.	$\forall x \in \mathbf{R}$ s.t. $x \neq 0\ \ \exists\ w \in \mathbf{R}$ s.t. $xw=1$	Existence of a Right Multiplicative Inverse
A9.	$x(y+z) = xy + xz$	Multiplication Distributes over Addition

There are other algebraic (field) properties which follow from these nine fundamental properties. Some of these additional properties (e.g., cancellation laws) are listed in high school algebra books and calculus texts.

ORDER PROPERTIES. There exists the subset $P = \mathbf{R}^+$ of positive real numbers that satisfies the following:

O1. $x,y \in P$ implies $x+y \in P$

O2. $x,y \in P$ implies $xy \in P$

O3. $x \in P$ implies $-x \notin P$

O4. $x \in \mathbf{R}$ implies exactly one of $x = 0$ or $x \in P$ or $-x \in P$ holds (trichotomy).

Note that the order properties involve the binary operations of addition and multiplication and are therefore linked to the field properties. There are other order properties which follow from the four fundamental properties. Some of these are listed in high school algebra books and calculus texts. The symbols $<$, $>$, $\leq$, and $\geq$ can be defined using the set P of positive numbers. The properties of these relations can then be established.

LEAST UPPER BOUND PROPERTY: The **least upper bound property** leads to the **completeness property** that assures us that the real number line has no holes. This is perhaps the most difficult concept to understand. It means that we must include the irrational numbers (e.g. π and $\sqrt{2}$) with the rational numbers (fractions) to obtain all of the real numbers.

DEFINITION. If S is a set of real numbers, we say that b is an **upper bound** for S if for each
$x \in S$ we have $x \leq b$. A number c is called a **least upper bound** for S if it is an upper bound for S and if $c \leq b$ for each upper bound b of S.

LEAST UPPER BOUND AXIOM. Every nonempty set S of real numbers that has an upper bound has a least upper bound.

ADDITIONAL ALGEBRAIC PROPERTIES AND FACTORING. Additional algebraic properties are useful in solving scalar equations in a real variable. These can be formulated as finding the **roots** of the **equation** $f(x) = 0$ (i.e., the **zeros** of f) where f is a real valued functions of a real variable. Operations on equations can be defined which result in **equivalent equations** (i.e., ones with the same solution set). These are called **Equivalent Equation Operations** or EEO's. An important property of **R** (and indeed for any field) states that if the product of two real numbers is zero, then at least one of them is zero. Thus if f is a polynomial that can be factored so that $f(x) = g(x)\, h(x) = 0$, then either $g(x) = 0$ or $h(x) = 0$ (or both since in logic we use the **inclusive or**). Since the degrees of g and h will both be less than that of f, this reduces a hard problem to two easier ones. Repeating the process to obtain the product of linear factors yields all of the zeros of f (i.e., the roots of the equation). The Fundamental Theorem of Algebra states that any polynomial over **C** can be factored into linear factors and that any polynomial over **R** can be factored into linear and quadratic factors. Although this can be done for many special cases, according to Galois, there is no finite process for doing this for an arbitrary polynomial of degree five or larger.

ADDITIONAL ORDER PROPERTIES. Often application problems require the solution of inequalities. The symbols $<$, $>$, $\leq$, and $\geq$ can be defined in terms of the set of positive numbers $P = \mathbf{R}^+ = \{x \in \mathbf{R}: x>0\}$ and the binary operation of addition. As with equalities, operations on inequalities can be developed that result in **equivalent inequalities** (i.e., ones that have the same solution set). These are called **Equivalent Inequality Operations** or EIO's.

OTHER ADDITIONAL ANALYTIC PROPERTIES. Interestingly, the algebraic operation of multiplication in **R** generalizes to the notion of **dot product** in $\mathbf{R}^3$ and **inner product** in $\mathbf{R}^n$. The inner product in $\mathbf{R}^n$ gives the notions of **norm**, **metric**, and **topology** in $\mathbf{R}^n$. In **R**, the norm of a number is just its absolute value, $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ from which we get a metric and a topology (this involves the concepts of **open** and **closed** sets in **R**. We also get the notions of **convergence** of an **infinite sequence**, a **Cauchy sequence**, and **completeness** (no holes). Recall that **R** is complete because of the **least upper bound** property, but **Q** is not. A Cauchy sequence will converge if the space is complete, but not necessarily if it has holes. For example, the Cauchy sequence 1.4, 1.41, 1.414,... converges to $\sqrt{2}$ in **R**, but does not converge in **Q**.

Handout #3 MORE ON THE FRAMEWORK FOR FIND PROBLEMS Prof. Moseley
Chap. 2

We extend our discussion of the **framework for find problems** (FFP's) by giving more examples illustrating the types of sets Σ and properties or conditions C that we can use, the number of solutions that the problem might have and possible techniques for solution. We have seen that Σ can be a number system such as **R** or **C** which are concrete examples of the abstract <u>algebraic structure</u> known as a **field**. Elements in a <u>field</u> are called **scalars** and equations involving such elements are called **scalar equations**. Since a solution of two equations in two unknowns, say x and y, is an ordered pair $[x,y]^T$, Σ can also be the set $\mathbf{R}^2 = \{[x,y]^T: x,y\in\mathbf{R}\}$ of all **ordered pairs** of real numbers (or elements in any field). Since solutions to problems could have any number of unknown variables, Σ can also be the set $\mathbf{R}^n = \{[x_1,x_2,...,x_n]^T: x_i\in\mathbf{R}$ for $i = 1, 2, ... , n\}$. For differential equations, the solution is a function so that Σ could be a **function space** like $C^1(I) = \{f:I \to \mathbf{R}$: f(x) and f′(x) are continuous on the interval I$\}$. All of the above are examples of **vector spaces**. Very often in science and engineering problems, Σ has the algebraic structure of (i.e., is an example of) a vector space.

A problem Prob(Σ,C) is **well-posed** in a set theoretic sense if it has exactly one solution. We give examples to show that all problems are not well-posed. Any number of solutions are possible. Recall that we use Prob(Σ,C) for the solution set defined implicitly by C, Soln(Σ,C) for the solution set defined explicitly, and $S = \{x\in\Sigma:C(x)\}$ for the solution set during the solution process.

<u>EXAMPLES</u>.

Prob($\mathbf{R}$,$x + 3 = 2$) =$\{x\in\mathbf{R}: x + 3 = 2\}$= S = $\{-1\}$= Soln($\mathbf{R}$,$x + 3 = 2$) well-posed
Prob($\mathbf{R}$,$x^2 - 4 = 0$) = $\{x\in\mathbf{R}: x^2 - 4 = 0\}$ = S = $\{2, -2\}$= Soln($\mathbf{R}$,$x^2 - 4 = 0$) two solutions, not well-posed
Prob($\mathbf{Q}$, $x^2 - 2 = 0$) = $\{x\in\mathbf{Q}: x^2 - 2 = 0\}$ = S = $\varnothing$ = Soln($\mathbf{Q}$,$x^2 - 2 = 0$) no solution, not well-posed
Prob($\mathbf{R}$, $x^2 = 2$) = $\{x\in\mathbf{R}: x^2 = 2\}$ = S = $\{\sqrt{2}, -\sqrt{2}\}$ = Soln($\mathbf{R}$, $x^2 = 2$) two solutions, not well-posed
Prob($\mathbf{R}$, $x^2 + 1 = 0$) = $\{x\in\mathbf{R}: x^2 + 1 = 0\}$ = S = $\varnothing$ = Soln($\mathbf{R}$, $x^2 + 1 = 0$) no solution, not well-posed
Prob($\mathbf{C}$, $x^2 + 1 = 0$) = $\{x\in\mathbf{C}: x^2 + 1 = 0\}$ = S = $\{i, -i\}$ = Soln($\mathbf{C}$, $x^2 + 1 = 0$) two solutions, not well-posed
Prob($\mathbf{R}$,$x + 3 < 2$) = $\{x\in\mathbf{R}: x + 3 < 2\}$ = S = $\{x\in\mathbf{R}: x < -1\}$ = Soln($\mathbf{R}$, $x + 3 < 2$) an infinite number of solutions, definitely not well-posed.

The solution process gets us from the implicit definition of S (which we call Prob(Σ,C)) on the left to the explicit description on the right (which we call Soln(Σ,C)). For equations this process often (but not always) consists of steps where we rewrite the given equation as an **equivalent equation** with the same solution set. However, other processes such as **factoring** and **theorems** such as "if the product of two numbers is zero, then one of the numbers must be zero" can be of use. Deciding exactly what we mean by an explicit solution to a problem is really part of the problem. If the problem is well posed, and the unique solution has a name, we want to know it. If we can show that there exists exactly one solution and it does not have a name, we can give it one. If Σ = **R** and the problem is well-posed, we may want a decimal approximation to the solution (i.e, the one element in the solution set). For **approximate solutions**, we need a notion of when two elements in a set are close together and when a sequence of approximate solutions converge to the **exact solution**. For scalars, the **absolute value of the difference** provides such a measure. This can be extended to finite dimensional vector equations as the **norm of the difference** of two vectors.

If we do not have a finite solution procedure, we wish to first establish that a problem is well-posed; that is, existence and uniqueness. To see the importance of showing existence, suppose

Prob($x \in$**R**, $x^2 = 2$ and $x>0$) = {$x \in$**R**: $x^2 = 2$ and $x>0$}= S = { $\sqrt{2}$ } = Soln($x \in$**R**, $x^2 = 2$ and $x>0$).

We have given a name of the element that we claim is the only element in the solution set. But how do we know there is such a real number and how can we find a decimal approximation to it. If it is a rational number, we want one of its "fraction" names. We see that showing that a problem is well-posed and finding its name or the name of a close by neighbor can be two different processes and either task (or both) could be called the **solution process**. We indicate how to show that Prob(**R**, $x^2 = 2$ and $x>0$} is well-posed so that there exists exactly one positive number whose square is two and we may then call it $\sqrt{2}$. Obviously there is a process to obtain an approximation to solution of $\sqrt{2}$ that your calculator has been programed to do when you punch the correct buttons. Your calculator "knows" which real numbers have square roots and how to compute a rational approximation for a positive square root. To show that Prob(**R**, $x^2 = 2$ and $x>0$} is well-posed, we need the axiomatic properties of **R** previously given. If you are interested in a proof of existence and uniqueness of the solution to this problem, you might wish to reread these axiomatic properties before reading the proof outlined in the next two paragraphs.

It can be shown that S = {$x \in$**R**: $x^2 = 2$ and $x>0$} = $S_1 \cap S_2$ = $S_3 \cap S_4 \cap S_2$ where S_1 = {$x \in$**R**: $x^2 = 2$ }, S_2 = {$x \in$**R**: $x>0$}, S_3 = {$x \in$**R**: $x^2 \geq 2$ }, and S_4 = {$x \in$**R**: $x^2 \leq 2$ }. That $S_1 = S_3 \cap S_4$ can be shown using the **trichotomy** property of **R** (an order property) which asserts that for any $a \in$**R** we have exactly one of the following 1) $a = 0$, $a < 0$, or $a > 0$.

Since for all $x \in S_4$, $x \leq 5$ we have that S_4 has an upper bound. Since $0^2 = 0 \leq 2$, $0 \in S_4$ so that S_4 is not empty. Hence by the **least upper bound axiom**, S_4 has a least upper bound. Thus it exists whether we can calculate it or not and we call it $\sqrt{2}$. If we can show $\sqrt{2} \in S_3$ and $\sqrt{2} \in S_2$, we have existence of a solution. If we assume that there is another solution, say $s \in S$ and can show that $s = \sqrt{2}$, we have uniqueness. This helps explain why the solution process usually implies uniqueness (if indeed the solution is unique). This is totally independent of having a process for computing a rational approximation for $\sqrt{2}$. But we should show (or believe someone has shown) that there exists a unique solution to our problem before trying to compute an approximation to it.

To solve $z^2 + 1 = 0$ we "invent" the number i with the defining property: $i^2 = -1$. We then "define" the set of complex numbers as $\mathbf{C} = \{x+iy : x,y \in \mathbf{R}\}$. The notation $z = x+iy$ is known as the **Euler form** of z. Since we cannot really add x and i y, the set of complex numbers **C** is more rigorously defined using the concept of **ordered pair** as $\mathbf{C} = \{(x,y) : x,y \in \mathbf{R}\}$. $z = (x,y)$ is the **Hamilton form** of z. Thus the complex numbers can be identified with the points in the **Cartesian Product** $\mathbf{R}\times\mathbf{R} = \mathbf{R}^2$ (i.e., geometrically with points in the "complex" plane).

ARITHMETIC OF COMPLEX NUMBERS. **Addition** of complex numbers is similar to addition of vectors in a plane (i.e., in $\mathbf{R}^2$). If $z_1 = x_1+iy$, and $z_2 = x_2+iy_2$, then $z_1+z_2 =_{df} (x_1+x_2) + i(y_1+y_2)$. (e.g. $(3+2i) + (4-7i) = 7-5i$). However, there is nothing comparable to **multiplication** of complex numbers for vectors in the plane. If $z_1 = x_1+iy$, and $z_2 = x_2+iy_2$, then $z_1z_2 =_{df} (x_1x_2-y_1y_2) + i(x_1y_2+x_2y_2)$. Using these definitions, the nine properties of addition and multiplication in the definition of an **abstract algebraic field** can be proved so that the system $(\mathbf{C}, +, \cdot, 0, 1)$ is an abstract algebraic field. Computation of the product of two complex numbers is made easy using your knowledge of the algebera of **R**, the FOIL (first, outer, inner, last) method, and the defining property $i^2 = -1$:
$(x_1+iy_1)(x_2+iy_2) = x_1x_2+x_1iy_2 + iy_1x_2 + i^2y_1y_2 = x_1x_2 + i\,(x_1y_2 + y_1x_2) - y_1y_2 = (x_1x_2-y_1y_2) + i(x_1y_2+x_2y_1)$.
This makes evaluation of polynomial functions easy.

EXAMPLE. If $f(z) = (3 + 2i) + (2 + i)z + z^2$, then $f(1+i) = (3 + 2i) + (2 + i)(1 + i) + (1 + i)^2$
$= (3 + 2i) + (2 + 3i + i^2) + (1 + 2i + i^2) \quad = (3 + 2i) + (2 + 3i - 1) + (1 + 2i - 1) = 4 + 7i$

If $z = x+iy$, then the **complex conjugate** of z is given by $\bar{z} = x - iy$. Also the **magnitude** or **absolute value** of z is $|z| = \sqrt{x^2+y^2}$ (i.e., the distance to the origin in the complex plane).

THEOREM #1. If $z, z_1, z_2 \in \mathbf{C}$, then a) $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$, b) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$, c) $|z|^2 = z\,\bar{z}$, d) $\bar{\bar{z}} = z$.

Proof. We show $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$, using the standard procedure for proving identities in the STATEMENT/REASON format starting with one side and ending with the other, justifying every step. The proofs of the other identities are left as exercises. Since $z_1, z_2 \in \mathbf{C}$, there exists $x_1, y_1, x_2, y_2 \in \mathbf{R}$ such that $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$. Thus

STATEMENT	REASON
$\overline{z_1+z_2} = \overline{(x_1+iy_1)+(x_2+iy_2)}$	Definition of z_1 and z_2.
$= \overline{(x_1+x_2)+i(y_1+y_2)}$	Definition of addition of z_1 and z_2.
$= (x_1+x_1) - i\,(y_1+y_2)$	Definition of complex conjugate.
$= (x_1 - i\,y_1) + (x_2 + i\,y_2)$	Definition of addition of complex numbers.
$= \overline{z_1} + \overline{z_2}$	Definition of complex conjugate.

Hence for any complex numbers, $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$, we have , $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
Q.E.D.

<u>REPRESENTATIONS</u>. Since each complex number can be associated with a point in the (complex) plane, in addition to the **rectangular representation** given above, complex numbers can be represented using **polar coordinates**.
$z = x + iy = r\cos\theta + i\,r\sin\theta = r(\cos\theta + i\sin\theta) = r\,\underline{\angle\,\theta}$ (polar representation)
Note that $r = |z|$. You should be able to convert from Polar to Rectangular form using the relations
$x = r\cos\theta$ and $y = r\sin\theta$ and from Rectangular to Polar form using $r^2 = x^2 + y^2$ and $\tan\theta = y/x$ and knowledge of the quadrant θ is in. Often, it is helpful to sketch the number in the complex plane.

<u>EXAMPLE</u> Convert $z = 2\,\angle\pi/4$ to rectangular form.
<u>Solution</u>. $x = r\cos\theta = 2(\sqrt{2}/2) = \sqrt{2}$, $y = r\sin\theta = 2(\sqrt{2}/2) = \sqrt{2}$ so that $z = \sqrt{2} + \sqrt{2}\,i$

<u>EXAMPLE</u> Convert $z = 1 + \sqrt{3}\,i$ to polar form.
<u>Solution:</u> First sketch the number in the complex plane.

$\theta = \pi/3$ $\quad z = 2(\cos(\pi/3) + i\sin(\pi/3))$
$r = \sqrt{1+3} = 2$ $\quad = 2\,\underline{\angle\pi/3}$

$\sqrt{3}$ | $\cdot\ (1, \sqrt{3})$
| /
|/ $\theta = \pi/3$ (60°)
|———
1 $\quad r = 2$

<u>EXAMPLE</u> If $z_1 = 3 + i$ and $z_2 = 1 + 2i$, then in rectangular form.

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2} = \frac{(3+i)(1+2i)}{(1+2^2)} = \frac{(3+6i+i+2i^2)}{(1+4)} = \frac{(3-2+7i)}{5} = \frac{1}{5} + \frac{7}{5}i$$

<u>EXAMPLE</u> If $f(z) = \dfrac{z+(3+i)}{z+(2+i)}$, then f(4+i). $= \dfrac{(4+i)+(3+i)}{(4+i)+(2+i)} = \dfrac{(7+2i)}{(6+3i)}\;\dfrac{(6-3i)}{(6-3i)} = \dfrac{16}{15} - \dfrac{1}{5}i$

<u>THEOREM #2</u>. If $z_1 = r_1\,\underline{\angle\theta_1}$ and $z_2 = r_2\,\underline{\angle\theta_2}$. Then a) $z_1 z_2 = r_1 r_2\,\underline{\angle\theta_1 + \theta_2}$, b) If $z_2 \neq 0$, then $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\,\underline{\angle\theta_1 - \theta_2}$, c) $z_1^{\,2} = r_1^{\,2}\,\underline{\angle 2\theta_1}$, d) $z_1^{\,n} = r_1^{\,n}\,\underline{\angle n\theta_1}$.

<u>EULER'S FORMULA</u>. By definition $e^{i\theta} =_{df} \cos\theta + i\sin\theta$. This gives another way to write complex numbers in polar form.

$z = 1 + i\sqrt{3} = 2(\cos\pi/3 + i\sin\pi/3) = 2\,\underline{\angle\pi/3} = 2\,e^{i\pi/3}$
$z = \sqrt{2} + i\sqrt{2} = 2(\cos\pi/4 + i\sin\pi/4) = 2\,\underline{\angle\pi/4} = 2\,e^{i\pi/4}$

Euler's formula allows the extension of <u>exponential</u>, <u>logarithmic</u>, and <u>trigonometric</u> functions to complex numbers and that the standard properties for these functions still hold. It allows you to determine what these extensions should be and to evaluate these functions for all complex numbers.

EXAMPLE. If $f(z) = (2 + i)\, e^{(1+i)z}$, find $f(1 + i)$.

Solution. First $(1 + i)(1 + i) = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$. Hence

$$f(1 + i) = (2 + i)\, e^{2i} = (2 + i)(\cos 2 + i \sin 2) = 2 \cos 2 + i(\cos 2 + 2 \sin 2) + i^2 \sin 2$$

$$= 2 \cos 2 - \sin 2 + i(\cos 2 + 2 \sin 2) \qquad \text{(exact answer)}$$

$$\approx -1.7415911 + i(1.4024480) \qquad \text{(approximate answer)}$$

Handout #5 THE NEED FOR UNDERSTANDING THE THEORY IN Prof. Moseley
Chap. 2 MATHEMATICAL PROBLEM SOLVING

Our objective is not to just simply memorize methods for solving some set of evaluation problems, but to also understand **why we use the methods we choose**. That is, we not only want to learn methods and be able to apply them to problems where we are told they work (i.e., be trained or train ourselves to use these methods), but also to know what methods apply to what problems and to understand why these methods work and when and how they can be extended to other problems. That is, we wish to **understand the theory** behind the methods. We, as engineers, scientists, and applied mathematicians are motivated by the fact that equations (algebraic and differential) are used as mathematical models of scientific and other phenomena, particularly systems that vary with time and space. To understand how to solve equations, we must **understand the theory** behind the methods used.

Most, if not all, of the problems you have solved so far in mathematics have been of the evaluate or locate (find) type (see the first handout in this chapter). For **evaluation problems**, you learned (i.e., were trained in or trained yourself to use) well defined algorithms or **computational skills** such as addition, multiplication, raising a number to a power, extraction of roots, and evaluation of algebraic functions, finding a derivative or evaluating an integral. These were learned not only for solving everyday problems in life (e.g., balancing your check book) but also as tools for solving locate or find problems. For **locate problems** you were asked to find all objects (e.g., numbers) satisfying a given **property** or **condition** (e.g., an algebraic equation). A fundamental "plan of attack" or philosophy used for such problems is to reformulate the problem as an **equivalent problem** (e.g., an equivalent algebraic equation) that has the same **solution set**. This process is repeated until a form of the problem is found where the set of solutions is easy to determine (i.e., has a more explicit description). However, the exact steps in the solution process are not preordained, but are instead left up to the problem solver. (The fun is to see who can find the shortest route to the answer.)

For an algebraic equation such as $3x - 2 = 7$ that has exactly one solution, the idea is to use **Equivalent Equation Operations** to isolate, if possible, the **unknown number** on one side of the equation. The value on the other side is then the (unique) solution. But how do we know what operations yield equivalent equations? Theory is needed. If the number of solutions is finite (and small), **an explicit description of the solution set** consisting of the (names of) the solutions should be found. This same idea of isolating the unknown can be used for solving first order linear ODE's where calculus is used to isolate the **unknown function** on one side of the equation. However, not just one solution is obtained. The **solution set** for the ODE consists of an **infinite number of functions** parameterized by an arbitrary integration constant.

The fundamental philosophy of **reformulating the problem** not only applies to equations where there is only one solution but extends to problems such as $x^2 + 3x + 2 = 0$ where there are two solutions and to inequalities such as $3x - 2 < 7$ where there are an infinite number of solutions. However, more theory (i.e., more properties of **R**) and/or a clearer definition of what is meant by "**an explicit description of the solution set**" are needed to effect a solution algorithm. For the problem $3x - 2 < 7$, the solution set is $S = \{x\in\mathbf{R}: 3x - 2 < 7\} = \{x\in\mathbf{R}: x < 3\}$ so that $x < 3$ is a more explicit description of S then $3x - 2 < 7$ since the unknown has been isolated.

The fundamental philosophy of **reformulating the problem** to get a more **explicit description of the solution set** applies to any problem where it is not easy to precisely identify all of the elements in a predetermined set Σ that satisfy a given property (e.g., an equation or an inequality). If the philosophy succeeds, an explicit (or at least a more explicit) description of the solution set is obtained. If all steps are reversible, the **solution process** gives the **solution set** exactly. However, some **equation operations** such as "squaring both sides of the equation" can introduce **extraneous roots** i.e., result in a new problem whose solution set includes other elements in addition to solutions of the original problem. However, if the new problem (and hence the old problem) has only a finite number of solutions and there is a process for checking solutions (e.g., an equation that you can substitute the solution into), these can be **checked individually** to see which are solutions and which are <u>extraneous</u>. In fact, since scalar algebraic equations are testable, solutions should always be checked to determine if errors have been made. Solutions to differential equations can likewise be checked.