

A SERIES OF CLASS NOTES  
TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS  
TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

REMEDIAL CLASS NOTES

A COLLECTION OF HANDOUTS FOR REMEDIATION

IN FUNDAMENTAL CONCEPTS FOR

PROBLEM SOLVING IN MATHEMATICS

# CHAPTER 1

## Some Fundamental Concepts in Mathematics

Handout 1. Informal Logic and Math Vocabulary

Handout 2. Sets and Number Systems

Handout 3. Functions, Ordered Pairs, Cartesian Products, and Binary Operations

Handout 4. One-to-One Correspondences,  $N^{\text{th}}$  Order Products, and Linear Orderings

Handout 5. Listings, Properties of Elements and Properties of Subsets

Handout 6. Axiomatic Systems, Cardinality, Induction, and Names for Numbers

The primary tools of mathematics are thought and expression. Ideas and language are often considered to be inseparable. A formal mathematical discourse begins with **undefined terms** (primitive “objects” or concepts) and **axioms** that are assumed to be true without proof. **Definitions** are given and **theorems** are proved using **deductive reasoning** to build the **mathematical theory**. In **formal logic**, a **formal proof** is a finite sequence of **statements** where every step is either (1) an axiom, (2) a definition, (3) a previously proven statement, or (4) the result of applying a **rule of inference**. The last statement in the sequence is the **theorem** which has been formally proved. However, mathematicians almost never write this type of formal proof as they are often long, tedious, and boring. They may write in a formal manner, but their logic and proofs, from a logician's point of view, are usually informal.

Most mathematicians believe that all classical mathematics can be built on the foundation laid by **logic** and **set theory**. Mathematical logic requires all **propositions** (statements) to be either **true** or **false**. **Compound propositions** are constructed using the **logical connectives** of “**and**”, “**or**”, “**not**”, “**implies**”, and “**if and only if**”. The **truth value** (i.e. either **true** or **false**) of an **open statement** (e.g. an equation) depends on the value chosen for the **variable(s)** in the statement. Finding the values in the **domain of definition** for the variable(s) (i.e., the **set** of allowed values) which make the statement true solves a **mathematical problem**.

**Formal logic and axiomatic set theory** are often needed for research in pure mathematics. However, at an elementary level, we can proceed more quickly if we rely on **informal logic** (i.e., common sense or rational thought) to determine the truth value of statements and to develop an intuitive understanding of the elementary concepts in **naive set theory**. After elementary set theory has been established, formal definitions in terms of sets of the concepts of **ordered pair**, **Cartesian product**, and **relation** can be given. Then the concepts of **function**, **linear ordering**, **partial ordering**, **equivalence relation**, **binary operation**, **one-to-one correspondence**, **finite sequence**, **cardinality**, **mathematical induction**, and the **number systems** can be established. We will not follow this order exactly since some concepts can be thought of as fundamental and most, if not all, you already know.

On the other hand (OTOH), we will give some formal definitions and will use the **language of mathematics**. The symbols in the following tables, if not already known, must be learned. They are standard and provide a short hand notation that helps avoid writer's cramp. However, care must be taken since, for example, the symbols for the quantifiers (i.e.  $\exists$  and  $\forall$ ) have a precise meaning in formal logic and are uncommon in other formal mathematical literature. We also use other acronyms such as LHS and RHS for the left and right hand sides of an equation.

## LOGIC

<u>Math Symbol</u>	<u>English Translation</u>
1. $\Rightarrow$	implies; if ____, then ____.: ____ is necessary for ____.
2. $\Leftarrow$	is implied by; __ only if __.; We have __, if __.; __ is sufficient for __.
3. iff	if and only if, $\Leftrightarrow$ , is equivalent to, is necessary and sufficient for
4. $\exists$	there exists, there is, for at least one, for some (existential quantifier)
5. $\forall$	for all, for each, for any (universal quantifier)
6. s.t., $\ni$	such that
7. Q.E.D. , ■	Quod Erat Demonstrandum (Latin Phrase meaning “which was to be demonstrated”). We are done with the proof.

## SET THEORY

<u>Math Symbol</u>	<u>English Translation</u>
1. $\in$	is an element of
2. $\subseteq$	is a subset of
3. $\cup$	unioned with
4. $\cap$	intersected with
5. $\emptyset, \phi$	the empty set

## NUMBER SYSTEMS

<u>Math Symbol</u>	<u>English Translation</u>
1. $\mathbf{N}$ (or $\mathbb{N}$ )	Natural numbers = $\{1, 2, 3, \dots\}$
2. $\mathbf{W}$	Whole numbers = $\{0, 1, 2, 3, \dots\}$
3. $\mathbf{Z, J}$	Integers = $\{\dots -2, -1, 0, 1, 2, \dots\}$
4. $\mathbf{Q}$	Rational Numbers
5. $\mathbf{R}$ (or $\mathbb{R}$ )	Real Numbers
6. $\mathbf{C}$ (or $\mathbb{C}$ )	Complex Numbers

## MISCELLANEOUS

<u>Math Symbol</u>	<u>English Translation</u>
1. LHS	Left Hand Side (of an equation)
2. RHS	Right Hand Side (of an equation)
3. OTOH	On the other hand
4. WOLOG	Without loss of generality
5. $\mathbf{R}$	One dimensional space or time
6. $\mathbf{R}^2$	The plane (two dimensional space)
7. $\mathbf{R}^3$	Three dimensional space
8. $C(I)$	Continuous functions on the interval I
9. $I = (a, b)$	The open interval $I \subseteq \mathbf{R}$ containing x such that $a < x < b$ .
10. $\bar{I} = [a, b]$	The closed interval $\bar{I} \subseteq \mathbf{R}$ containing x such that $a \leq x \leq b$

In axiomatic set theory, the notion of a **set** is considered to be a primitive concept. This means that it is not formally defined but provides a starting point on which other concepts can be built. The axioms of a particular mathematical theory assert the existence of the fundamental sets used for the theory and provide properties for the objects in these sets. Informally, we define a set as a collection of objects. The objects are called the **elements** of the set. Elements are the building blocks for sets and could be considered to be a fundamental concept. (In one axiomatic system, all elements are sets or classes of sets.) We could consider sets containing cars, sets containing houses, or sets containing people. However, we are mostly interested in sets containing mathematical objects such as numbers. To define a particular set of numbers (or cars, or houses, or people) the important thing is to be able to determine whether a given element (e.g., number) is or is not a member of the set. This may involve the **names** of identifiable elements in the set (e.g., the set containing 1, 5, and 7) or it may involve **properties** of the elements in the set (e.g., the set containing the even positive integers). A set with structure is often called a **space** although number systems are usually not referred to as spaces.

**ELEMENTARY SET DESCRIPTIONS AND NUMBER SYSTEMS.** To define a set which contains only a small number of elements, say, less than ten, we can list (the **names** of) the elements. Using standard notation, the set  $S$  consisting of the numbers 1, 7, and 10 can be defined by writing  $S = \{1,7,10\}$ . (This assumes previous knowledge of the numbers 1, 7, and 10. The finite sequence 1, 7, 10 is called a "**single**" **listing** of the elements in  $S$ .) Larger sets of numbers can often be clearly specified using the  $\dots$  (ellipsis) notation. For example,  $T = \{1,2,3,\dots,100\}$  clearly defines the set of integers between 1 and 100 (including 1 and 100). As was just done we can often describe sets of numbers using English. However, we prefer the "set builder" notation since, as we have just seen, English definitions can sometimes be ambiguous. To avoid the ambiguity more words had to be used. The "set builder" notation is clearer and shorter. The sequence 1, 2, 3, ..., 100 is a "single" **listing** of the elements in  $T$ . Although not usually done, we do allow "**multiple**" **listings** of elements so that  $\{1,2,2,3\} = \{1,2,3\}$ .

The  $\dots$  notation can also be used for **infinite sets**. For example,

$$\mathbf{N} = \{1,2,3,\dots\}$$

is a standard way of describing the set of **positive integers** (also called the **natural numbers** or **counting numbers**). Since we have not rigorously constructed  $\mathbf{N}$ , but rely on your previous knowledge of the counting numbers to interpret the meaning of the  $\dots$  notation, we use the word "describe" rather than the word "define". In some texts,  $\mathbf{N}$  denotes the **non-negative integers**, but to avoid confusion we let  $\mathbf{W} = \{0,1,2,3,\dots\}$  (whole numbers, as denoted in some elementary texts) be the set of non-negative integers. Also using set notation

$$\mathbf{Z} = \{\dots,-2,-1,0,1,2,\dots\}$$

describes the set of all **integers**. (Some texts use  $\mathbf{J}$  for the set of integers.) After several sets of

numbers have been established, set theory notation can be used to define (or describe) other sets. First we need a notation to indicate that an element (e.g., number) is a member of a set. We write  $3 \in \mathbf{N}$  to mean that the number 3 is an element of the set of positive integers. This translation may be shortened to read "3 is a positive integer". Having defined the use of the  $\in$  notation we can use it to define or describe more complicated sets of numbers. For example,

$$\mathbf{Q} = \{p/q : p, q \in \mathbf{Z} \text{ and } q \neq 0\}$$

describes the set of **rational numbers**. An English translation of this notation is "**Q** is the set of all possible quotients,  $p$  divided  $q$ , where  $p$  and  $q$  are integers and  $q \neq 0$ ". The colon is often translated as "such that" (a vertical line is sometimes used instead of a colon). Hence another translation might be "**Q** is the set of all numbers  $p$  divided by  $q$  such that  $p$  and  $q$  are integers and  $q \neq 0$ ". A shorter translation is "**Q** is the set of quotients of integers where the denominator is nonzero". This assumes knowledge of the algebraic structure of **Q** (i.e., the terms quotient and denominator). Thus, this is not a rigorous translation until we define this algebraic structure. Your previous knowledge of fractions fills the gap. The important thing is to clearly describe the set. Can the reader of the set description tell whether a given number is or is not in the set? Note that this description of **Q** really has **multiple listings**. For example, both  $3/4$  and  $6/8$  are in the set but are really two different names for the same element. Set descriptions often involve **properties** of the elements; that is, conditions that the elements must satisfy to be in the set (e.g.  $p \in \mathbf{Z}, q \neq 0$ ). If these statements are true (i.e., if the element has these properties), then the element is in the set. If (one of these is) false (i.e., if the element fails to have one of the properties), then the element is not in the set.

Unfortunately, the set of **real numbers** (the standard notation is **R**) is not easily described in terms of the rational numbers. Vaguely, the real numbers **R** consist of the rational numbers

and the **irrational numbers** (such as  $\sqrt{2}$  and  $\pi$ ). If we assume that we can decide if a number is a real number or not we can describe the **complex numbers** using **Euler's notation** as

$$\mathbf{C} = \{a+bi : a, b \in \mathbf{R}\}$$

where  $i = \sqrt{-1}$ . A more rigorous definition of **C** is as a set of **ordered pairs** of real numbers,  $\mathbf{C} = \{(a,b) : a, b \in \mathbf{R}\}$  since an ordered pair can be defined rigorously in terms of sets. For now, we will be content with the notation  $(a,b)$  and the "informal definition" that  $(a,b) = (c,d)$  if (and only if)  $a = c$  and  $b = d$ .

If all of the elements of set  $A$  are also elements of set  $B$ , we say  $A$  is a **subset** of  $B$  and write  $A \subseteq B$ . The notation  $A \subset B$  is used to mean subset in some texts, but we avoid this since it is used in other texts to mean proper subset ( $A$  is a **proper subset** of  $B$  if  $A \subseteq B$ , but  $A \neq B$ .) Since a set is defined by the elements it contains, we see that  $A = B$  if and only if  $A \subseteq B$  and  $B \subseteq A$ .

For our number systems, it is important to note that

$$\mathbf{N} \subseteq \mathbf{W} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R} \subseteq \mathbf{C}$$

and that these are all proper subsets. All of these sets of numbers have structure, particularly algebraic structure (e.g., addition and multiplication) and hence are spaces. Rigorous construction of the number systems requires much explanation and consumes volumes. Briefly, using formal logic, axiomatic set theory, definitions, and theorems, the **Peano Postulates** can be established. From these, **N** can be constructed. Then, in succession **W**, **Z**, **Q**, **R** and **C**.

**COMBINATIONS OF SETS.** Once we have a collection of sets defined (we can define any set we wish, not just the standard sets), we can combine these using the operations of union and intersection. The **union** of two sets is just the set containing all of the elements in either set. For example, if we define  $S = \{1,2,3\}$  and  $T = \{1,3,8\}$ , then the union of  $S$  and  $T$ , denoted by  $S \cup T$ , is the set of all elements that are in  $S$  **or**  $T$  (or both) and hence is the set  $\{1,2,3,8\}$ . Thus, given  $S$  and  $T$  defined as above (note that we have redefined  $S$  and  $T$  from what they were previously) we "compute"  $S \cup T = \{1,2,3,8\}$ . Note that in describing a set by listing it, we do not normally list an element twice. If the element is in the set, it is listed; if not, it is usually not listed. However, we do allow multiple listings so that  $\{1, 2, 2, 3\} = \{1, 2, 3\}$ . Also, the order in which the (names of the) elements are listed is not part of the definition of the set. Thus  $\{1,2,3,8\} = \{8,3,2,1\} = \{1,8,3,2\}$ . The **intersection** of two sets is the set consisting of the elements that are in both sets. The intersection of  $S$  and  $T$ , written  $S \cap T$  is thus the set that contains all elements in  $S$  **and**  $T$  and is "computed" as  $S \cap T = \{1,3\}$ . Note that the words **and** and **or** have become technical words and indeed have technical definitions in **formal logic** when forming **compound propositions**. Formal logic uses the **inclusive or** so that opening Door A or B does not preclude opening both doors. If we wish the **exclusive or**, we use additional words such as "Open Door A or B, but not both".

**THEOREMS AND PROOFS.** A theorem is simply an important fact worth remembering. A proof is simply a clear explanation of why a fact is true. Facts about small sets can be proved (or checked) by examining the elements in the set. To review (or introduce you to) the language of proofs, we state and prove one such elementary fact in the Theorem/Proof format.

**THEOREM.**  $\{1,4\} \subseteq \{1,2,3,4\}$ .

Proof. By the definition of subset, to show that  $A \subseteq B$ , we must show (or check) that all elements in  $A$  are also in  $B$ . Since the only elements in  $\{1,4\}$  are 1 and 4 and these are both in  $\{1,2,3,4\}$ , we see that  $\{1,4\} \subseteq \{1,2,3,4\}$  as was to be proved. Q.E.D.

To show that two sets are equal, we simply have to show that they have the same elements. This is easily done in two parts. Thus to show  $A = B$ , we show  $A \subseteq B$  and  $B \subseteq A$ .

**THE EMPTY SET, THE POWER SET AND THE UNIVERSAL SET.** It is convenient to have a notation for the "set" which has no elements. We refer to it as the **empty set** and denote it by  $\emptyset$ . There is only one empty set. If we let  $S = \{1,2,3\}$ ,  $T = \{1,3,8\}$ , and  $U = \{5,6,7\}$ , then  $S \cap U = \emptyset$ ,  $T \cap U = \emptyset$ ,  $(S \cup T) \cap U = \emptyset$ ,  $S \cap \emptyset = \emptyset$ , and  $S \cup \emptyset = S$ . In **axiomatic set theory**, the existence of  $\emptyset$  is simply postulated. Our informal definition will allow your intuition to

determine the truth value of statements involving  $\emptyset$ .

The **power set** of the set  $A$ , denoted by  $P(A)$ , is the set consisting of all of the subsets of  $A$ . Unfortunately, naive set theory contains paradoxes. Set theorists work hard to axiomatically construct a theory which avoids these paradoxes. (The discovery of these is sometimes referred to as the third crisis at the fundamentals of mathematics.) These may involve constructing bigger and bigger sets (e.g., by computing power sets of known infinite sets using the **Axiom of Choice** which states that an element can be selected from each set in a family of sets). We can avoid these difficulties by requiring in any given discussion the existence of a **universal set** or **universe**, denoted by  $U$ , which contains all of the elements under consideration. (A universal set with properties defined by axioms is called a space, e.g., a number system is a space although not usually referred to as such). We consider subsets of  $U$  and  $P(U)$ , but do not allow  $P(U)$  to be an element of  $U$ . The universal set is often not given explicitly. However, if we are careful about the elements, sets, and subsets that we do consider, we will not get into logical difficulty. The **complement of the set**  $A \subseteq U$ , denoted by  $\overline{A}$  or  $-A$  and read "A complement", is the set of all elements in the universal set  $U$  that are not in the set  $A$ .

**PROPERTIES OF UNION AND INTERSECTION.** The operations of union and intersection have a number of properties. We list only a few. Let  $A$ ,  $B$ , and  $C$  be sets, then

- |  |                             |
|--|-----------------------------|
| 1. $A \cup B = B \cup A$                   | Union is commutative        |
| 2. $(A \cup B) \cup C = A \cup (B \cup C)$ | Union is associative        |
| 3. $A \cap B = B \cap A$                   | Intersection is commutative |
| 4. $(A \cap B) \cap C = A \cap (B \cap C)$ | Intersection is associative |

Many properties of sets can be proved. Recall that to show that  $A \subseteq B$ , we must show (or check) that all elements in  $A$  are also in  $B$ . If the sets are not specified, but arbitrary, we must show that an arbitrary element in  $A$  is also in  $B$ .

**THEOREM.**  $A \subseteq A \cup B$ .

**Proof.** By the definition of subset, to show that  $A \subseteq B$ , we must show that an arbitrary element in  $A$  is also in  $B$ . Let  $x$  be an arbitrary element in  $A$ .

STATEMENT

$x \in A$

$x \in A$  or  $x \in B$

$x \in A \cup B$

REASON

Assumption (or Given).

Definition of the (inclusive) or.

Definition of union.

Since we have shown that an arbitrary element in  $A$  is also in  $A \cup B$ , by the definition of subset, we have that  $A \subseteq A \cup B$ . Q.E.D.

**THEOREM.** (1)  $B \subseteq A \cup B$ . (2) if  $A \subseteq B$ , then  $A = A \cap B$ . (3)  $A \cap B \subseteq A \cup B$ .

From high school, you should be familiar with the concepts of "set", "number", "order", "function", and "addition". In axiomatic set theory, the notion of a **set** is considered to be a primitive concept. It is not formally defined, but is given the informal definition of a collection of objects called the elements of the set. The concepts of ordered pair, Cartesian product, relation, finite sequence, linear ordering, function and binary operation can then be given formal definitions in terms of sets. However, often it is easier to think of a concept as also being a primitive using an informal definition. The informal definitions are sufficiently precise for our purposes and allow faster progress. We may give or focus on an informal definition that you are familiar with before or after we give the formal definition. It is important to know that a formal definition can be given in terms of sets. It is more important to have a good understanding of the concept which may be provided by the informal definition.

Although the notion of **order** can be defined in terms of sets, it can also be thought of as being fundamental. Before and after are natural spacial and temporal ideas so lining up elements may be thought of as primitive. Informally, an **ordered pair** from the sets A and B is an element of the form (a,b) where  $a \in A$  and  $b \in B$ . The ordered pair (a,b) equals the ordered pair (c,d) if (and only if)  $a = c$  and  $b = d$ . That is, order is important in determining equality whereas as sets,  $\{a,b\} = \{b,a\}$  since these sets have exactly the same elements. (The order we list them is not important. The set  $\{a,b\}$  is sometimes called an **unordered pair**.) In terms of sets, (a,b) may be defined formally as  $(a,b) = \{\{a\}, \{a,b\}\}$ . After an ordered pair has been defined, we may formally define the **Cartesian product** of the sets A and B as the set of ordered pairs  $A \times B = \{(x,y) : x \in A \text{ and } y \in B\}$ . The Cartesian product of a set A with itself is  $A \times A = \{(x,y) : x,y \in A\}$  which we denote by  $A^2$ . We define formally a **relation R** between the sets A and B as a subset of  $A \times B$ . A **relation R on A** is a subset of  $A^2$ . The notation  $xRy$  (x is related to y) means  $(x,y) \in R$ . Familiar examples of relations are  $<$  (less than) as well as  $>$ ,  $\leq$ , and  $\geq$ . Some properties of relations are given in the following definition.

**DEFINITION.** Let R be a relation on A. R **passes the vertical line test** if  $\forall (x_1,y_1), (x_2,y_2) \in R$  ( $x_1Ry_1, x_2Ry_2$ ) we have  $x_1 = x_2$  implies  $y_1 = y_2$ . R is **reflexive** if  $\forall x \in A, (x,x) \in R$  ( $xRx$ ). R is **antisymmetric**. if  $\forall x,y \in A, (x,y) \in R$  and  $(y,x) \in R$  implies  $x = y$  ( $xRy$  and  $yRx$  implies  $x = y$ ). R is **transitive** if  $\forall x,y,z \in A, (x,y) \in R$  and  $(y,z) \in R$  implies  $(x,z) \in R$  ( $xRy$  and  $yRz$  implies  $xRz$ ). We say (all) elements in A are **comparable** (using R) if  $\forall x,y \in A, (x,y) \in R$  or  $(y,x) \in R$  ( $xRy$  or  $yRx$ ). R is **symmetric** if  $\forall x,y \in A, (x,y) \in R$  implies  $(y,x) \in R$  ( $xRy$  implies  $yRx$ ).

Defined formally in terms of sets, a **function** is a relation that passes the vertical line test.

The Cartesian product of  $A^2$  with A is  $A^2 \times A = \{(x,y,z) : x,y,z \in A\}$  and the Cartesian product of A with  $A^2$  is  $A \times A^2 = \{(x,(y,z)) : x,y,z \in A\}$ . Although these sets are technically different, there is an "obvious" **one-to-one correspondence** between these sets and between each of them and the set of **ordered triples** defined informally as  $A \times A \times A = \{(x,y,z) : x,y,z \in A\}$ . Normally we make no distinction between these three sets and denote all three by  $A^3$ . By using **mathematical induction**, we also define the **n<sup>th</sup> order Cartesian product** of A informally as



$A^n = \underbrace{A \times A \times \dots \times A}_{n\text{-terms}} = \{(x_1, x_2, \dots, x_n) : x_i \in A \text{ for } i = 1, 2, \dots, n\}$ . We may refer to the elements

in  $A^n$  as (row) vectors and denote them using  $\vec{x} = (x_1, x_2, \dots, x_n)$ . If  $\vec{x} = (x_1, x_2, \dots, x_n)$ ,  $\vec{y} = (y_1, y_2, \dots, y_n) \in A^n$ , then  $\vec{x} = \vec{y}$  if (and only if)  $x_1=y_1, x_2=y_2, \dots, x_n=y_n$ .

Informally, a **finite sequence** of length  $n$  has the form  $x_1, x_2, \dots, x_n$ . We may identify the set of **finite sequences of elements in  $A$  having length  $n$**  with the set  $A^n$ . Thus two sequences  $x_1, x_2, \dots, x_n$  and  $y_1, y_2, \dots, y_n$  are equal if (and only if)  $x_1=y_1, x_2=y_2, \dots, x_n=y_n$  (or  $x_i = y_i$  for  $i = 1, 2, \dots, n$ ). We may think of the sequence  $x_1, x_2, \dots, x_n$  as a **linear ordering** of the elements in the set  $A = \{x_1, x_2, \dots, x_n\}$  if we have assumed a **"single" listing** of the elements in  $A$  (i.e.,  $x_i = x_j$  implies  $i = j$ ).

To develop an intuitive understanding of the concept of function, rather than use the formal definition, we use the informal definition of a function as a **rule of correspondence** between two sets. We may then define the **graph** of a function as a specific set of **ordered pairs** in a **Cartesian product**. (You should make a distinction between this definition of the graph of a function as a set and the "picture" graph of a real function of a real variable that is a geometrical interpretation of this set that we draw on a Cartesian coordinate system.) Since this definition of a graph of a function is really the formal definition of a function, there is a **one-to-one correspondence** between **functions** and their **graphs**. That is, the graph of a function is really the formal definition of a function as a set as it defines a function in terms of sets. Having defined functions in terms of sets, we can then (formally) define a **binary operation** such as addition or multiplication as a function on the Cartesian product of a set with itself. Hence we can define an ordered pair, Cartesian product, function and binary operation in terms of sets. This is in agreement with the notion that all of mathematics can be built on the foundation of formal logic and axiomatic set theory. However, in practice, it is easier to think of set, ordered pair and function separately using informal definitions. The informal definition of a function is sufficiently precise and so important as to warrant being repeated as a (formal) definition.

**DEFINITION.** A **function** is a rule of correspondence which assigns to each element in a first set (called the **domain** of the function) exactly one element in a second set (called the **co-domain** of the function).

We can consider functions from any set to any other set. You are familiar with real valued functions of a real variable, that is, functions that map a subset of the real numbers **R** to the real numbers **R**. It is important to note that to define a function, we must first define two sets, the domain and the co-domain, before giving the rule of correspondence. Thus, these two sets are part of the definition of a function. A function is not completely defined unless both of these sets have been specified. (Often in high school algebra texts these are omitted since the codomain is almost always **R** and the "natural" domain is the subset of **R** where the formula or rule which defines the function is defined.)

Often we denote the function or rule by  $f$ . If  $x$  is any element in the domain, then  $y = f(x)$  indicates the element  $y$  in the co-domain that the rule defined by  $f$  associates with the element  $x$  in the domain. The **independent variable** is  $x$  and the **dependent variable** is  $y$ . We use the notations  $f: A \rightarrow B$  and  $A \xrightarrow{f} B$  to indicate that  $A$  is the domain and  $B$  is the co-domain of the function  $f$ . We may also denote the domain of  $f$  by  $D_f$  or by  $D(f)$ . The **graph** of  $f: A \rightarrow B$  is the set  $G = \{ (x, f(x)) \in A \times B : x \in A \}$ . It is a subset of the Cartesian product  $A \times B = \{ (x, y) : x \in A \text{ and } y \in B \}$ . To obtain a rigorous definition of a function in terms of sets, we can define a function to be its graph. Make a distinction between the graph of a function as a set and the geometric pictures (which we will call "picture" graphs) of real valued functions of a real variable that you have drawn on paper and the chalk board using **Cartesian coordinates**. Formally, a **relation** on  $A \times B$  that passes the **vertical line test** is a **function**.

If  $f: A \rightarrow B$ , then the **range** of  $f$  is the set  $R_f = \{y \in B: \exists x \in A \text{ s.t. } f(x) = y\}$  which is a subset of the codomain. Thus the domain of a function is the set of things that get mapped and the range is the set of things that get mapped into. That is, the range is the set of things in the codomain (target set) for which there exist an element in the domain (quiver set) that gets mapped (or shot) into those things (ie., the range is the things that actually get hit).

To develop an intuitive feel for the properties of functions, we will continue to use the informal definition of a function as a rule of correspondence from the domain to the codomain. We also accept the existence of **binary operations** such as addition, subtraction, multiplication and division as part of the algebraic structure of number systems and use formulas such as  $f(x) = x^2 + 3x + 5$  to define the rule of correspondence for some functions.

The terms “**mapping**”, “**function**”, “**operator**”, “**transform**”, and “**transformation**” all have essentially the same meaning. Different words are used in different contexts to indicate the nature of the **domain** and **codomain** of the mappings under consideration. Recall that the word function is usually used for mappings from  $\mathbf{R}$  to  $\mathbf{R}$ . The word operator is used for a mapping from a set of functions to another set of functions. The terms transformation and transform are often used for mappings from a set back to the same set. An element transforms (or "morphs") from one form (element) in the set to another. We use the notation  $F(A,B) = \{f:A \rightarrow B\}$  to denote the set of all functions from  $A$  to  $B$ . Thus  $F(\mathbf{R},\mathbf{R})$  is the set of all real valued function of a real variable whose domain is  $\mathbf{R}$ .

DEFINITION. The **graph** of  $f:A \rightarrow B$  is the set  $G = \{ (x,f(x)) \in A \times B: x \in A \}$ . It is a relation between  $A$  and  $B$  (i.e., a subset of the Cartesian product  $A \times B = \{(x,y): x \in A \text{ and } y \in B\}$ ) such that if  $(x_1,y_1), (x_2,y_2) \in A \times B$ , then  $x_1 = x_2$  implies  $y_1 = y_2$ . (For real valued functions of a real variable, this is called the vertical line test.)

To obtain a rigorous definition of a function in terms of sets, we have formally defined a function to be its graph.

Finally, we give a formal definition of one-to-one correspondence in terms of functions.

DEFINITION. Let  $f: X \rightarrow Y$ . The function  $f$  is said to be **one-to-one (injective)** if  $f(x_1) = f(x_2)$  implies  $x_1 = x_2$ . (This is the contra positive of the statement, if  $x_1 \neq x_2$ , then  $f(x_1) \neq f(x_2)$ . That is, distinct values of  $x$  in  $X$  get mapped to distinct values of  $y$  in  $Y$ . Recall that the negation of a negative is the positive. For real valued functions of a real variable, this is called the horizontal line test.) The function  $f$  is said to be **onto (surjective)** if  $R(f) = Y$ . (That is,  $f$  is onto if the range is the entire co-domain.) If  $f$  is both 1-1 and onto, then it is said to be **bijective** or to form a **one-to-one correspondence** between the domain and the co-domain.

Recall that  $a \in A$  can be paired with  $b \in B$  using the notation  $(a,b) \in A \times B$ . We could just as well have used  $[a,b]$ ,  $\begin{bmatrix} a \\ b \end{bmatrix} = [a,b]^T$ , or  $\langle a,b \rangle$  for an **ordered pair**. The concept of a **one-to-one correspondence** has been defined in terms of functions. However, the concept of **pairing** each and every element in one set with one and only one element in another set so that every element in the second set has a mate may be viewed as fundamental or primitive, just like that of **set** and **function**. Shepards in olden times kept tract of their sheep by putting them in one-to-one correspondence with a pile of rocks without necessarily knowing how to count their sheep or how many sheep they had.

Let  $n$  be an arbitrary number in  $\mathbf{N}$  and  $\mathbf{N}_n = \{1, 2, 3, \dots, n\}$  be the **finite set** consisting of the first  $n$  numbers in  $\mathbf{N}$ . The concept of **cardinality** answers the question “How many elements does a set have?”. This involves ideas at the very foundation of mathematics. If, for some  $n \in \mathbf{N}$ , there exists a one-to-one correspondence between a set  $A$  and the set  $\mathbf{N}_n$  (i.e., we may pair the elements in  $A$  with the numbers in  $\mathbf{N}_n$ ), then  $A$  has cardinality  $n$  and  $A$  is said to be **finite**. If we remove the elements in  $A$  one at a time and number them until all elements are gone, we obtain a **listing** (i.e., the finite sequence  $\{a_i\}_{i=1}^n = a_1, a_2, \dots, a_n$ ) of the elements in  $A$ ... Since every element receives a number, the mapping is onto. Since  $a_i = a_j$  implies  $i = j$  the mapping is also one-to-one, and we have a **single listing**.

When there is an “obvious” one-to-one correspondence between two sets (e.g., when they are finite, have the same number of elements, and are linearly ordered), we may or may not make a distinction between the two sets. There is an obvious one-to-one correspondence between  $\mathbf{N}_5 = \{1,2,3,4,5\}$  and  $S = \{a,b,c,d,e\}$ , but we normally consider these sets to be different sets. Hence having the same cardinality and some of the same structure (both  $\mathbf{N}_5$  and  $S$  are ordered) is not sufficient to identify them as the same set. On the other hand (OTOH), there are sets where we normally make no distinction. Recall that the sets  $A^2 \times A = \{(x,y),z\} : x,y,z \in A\}$  and  $A \times A^2 = \{(x,(y,z)) : x,y,z \in A\}$  are in an “obvious” one-to-one correspondence with the set of **ordered triples**  $A \times A \times A = \{(x,y,z) : x,y,z \in A\}$ . Often we make no distinction between these three sets and denote all three by  $A^3$ . We may consider elements in corresponding sets to be the same element.

Recall that we define the  **$n^{\text{th}}$  order Cartesian product** of  $A$  as  $A^n = \underbrace{A \times A \times \dots \times A}_{n \text{ terms}} = \{(a_1, a_2, \dots, a_n) : a_i \in A \text{ for } i = 1, 2, \dots, n\}$  as the set of **finite sequences** of elements in  $A$  having length  $n$ . We may also identify finite sequences of length  $n$  without parenthesis (that is, those of the form:  $a_1, a_2, \dots, a_n$ ) with the set  $A^n$ . Regardless, the two sequences  $(a_1, a_2, \dots, a_n)$  and  $(b_1, b_2, \dots, b_n)$  are equal if (and only if)  $a_1 = b_1, a_2 = b_2, \dots, a_n = b_n$ . Thus we may think of the finite sequence  $(a_1, a_2, \dots, a_n)$  as a **linear ordering** of the elements in  $\{a_1, a_2, \dots, a_n\}$  if we assume a single listing.

Intuitively,  $\mathbf{N}$  is the “smallest” infinite set in mathematics. Thus  $\mathbf{N}$  is useful not only as a number system, but also very important in establishing fundamental ideas about **infinity**. The idea that although the set  $\mathbf{N}_n = \{1, 2, 3, \dots, n\}$  is finite, we can always add one more is called

“potential infinity”. When we write  $N = \{1,2,3,\dots\}$  we have an "actually infinite set". If there exists a one-to-one correspondence between a set A and the infinite set  $N = \{1,2,3,\dots\}$ , then A has cardinality  $\aleph_0$  ( $\aleph$ , aleph, is the first letter of the Hebrew alphabet) and A is said to be **countably infinite**. A set is **countable** if it is finite or countably infinite. A one-to-one correspondence between the infinite sets  $N$  and  $Z$  can be constructed as follows:

$N$	1	2	3	4	5	6	7	8	9	10	...
$Z$	0	1	-1	2	-2	3	-3	4	-4	5	...

that is, by the function  $f:N \rightarrow Z$  defined by

$$f(n) = \begin{cases} n/2 & \text{for } n = 0, 2, 4, \dots \\ -(n-1)/2 & \text{for } n = 1, 3, 5, \dots \end{cases} . \text{ Hence } Z \text{ is countably infinite. However, } N \text{ and } Z \text{ have}$$

different structures (addition and multiplication) so that we always consider them to be different sets.

There is also a one-to-one correspondence between  $Q$  and  $N$  so that  $Q$  is also countably infinite. It has been shown that there exists no one-to-one correspondence between  $R$  and  $N$ , so that  $R$  is **uncountably infinite**. Its cardinality is often denoted by  $c$  (for continuum). How many mathematical infinities are there? Well there are of course an infinite number of such mathematical infinities.

Listings need not be finite. A countable listing is given by the infinite sequence  $\{a_i\}_{i=1}^{\infty} = a_1, a_2, \dots, a_n, \dots$ . In fact, listings need not be countable, linearly ordered, or one-to-one. They need only be onto. We use  $\{a_x\}_{x \in \Omega}$  where  $a_x$  defines a map from a set  $\Omega$  with known structure to  $A$  that is onto to denote such a listing. If the mapping is also one-to-one so that it is a one-to-one correspondence between  $\Omega$  and  $A$ , then we have a **single listing** of the elements in  $A$ . **Linear orderings** need not be countable.

DEFINITION. A relation  $\leq$  on a nonempty set  $A$  is a **linear ordering** of  $A$  if

- a)  $\forall x \in A, (x,x) \in \leq$  ( $x \leq x$ ). ( $\leq$  is reflexive.)
- b)  $\forall x, y \in A, (x,y) \in \leq$  and  $(y,x) \in \leq$  implies  $x=y$  ( $x \leq y$  and  $y \leq x$  implies  $x=y$ ). ( $\leq$  is antisymmetric.)
- c)  $\forall x, y, z \in A, (x,y) \in \leq$  and  $(y,z) \in \leq$  implies  $x \leq z$  ( $x \leq y$  and  $y \leq z$  implies  $x \leq z$ ). ( $\leq$  is transitive.)
- d)  $\forall x, y \in A, (x,y) \in \leq$  or  $(y,x) \in \leq$  ( $x \leq y$  or  $y \leq x$ ). (All  $x$  and  $y$  in  $A$  are comparable).

The number systems  $N, W, Z, Q$ , and  $R$  are all linearly ordered.  $C$  is not.

THEOREM. Let  $\leq$  be a linear ordering that contains more than one element. If  $(x,y) \in \leq$  (that is,  $x \leq y$ ) and  $x \neq y$ , then either  $(x,y) \in \leq$  or  $(y,x) \in \leq$ , but not both (either  $x \leq y$  or  $y \leq x$ , but not both).

If  $x \leq y$  and  $x \neq y$ , we write  $x < y$ . The mutual exclusiveness of the three possibilities:  $x < y$ ,  $y < x$ , or  $x = y$  of a linear ordering is called the **trichotomy property**. A relation on a set that is only reflexive, antisymmetric and transitive is called a **partial ordering**. The subsets of a universal set  $U$  (i.e.,  $P(U)$ ) are partially ordered by set inclusion. That is,  $A, B \in P(U)$  are related if  $A \subseteq B$ .

Let  $U$  be a universal set for an axiomatic system (i.e., let  $U$  be a space). Let  $A \subseteq U$  so that  $A \in P(U)$ . If  $A$  is finite, we may put its elements in one-to-one correspondence with the set  $N_n = \{1, 2, 3, \dots, n\}$ . That is we may obtain a listing (or finite sequence)  $\{a_i\}_{i=1}^n = a_1, a_2, \dots, a_n$ . If  $a_i = a_j$  implies  $i = j$ , then we have a "single listing". Listings need not be finite. A countable listing is given by the infinite sequence  $\{a_i\}_{i=1}^{\infty} = a_1, a_2, \dots, a_n, \dots$ . In fact, listings need not be countable. We use  $\{a_x\}_{x \in \Omega}$  where  $a_x$  defines a map from a known set  $\Omega$  to  $A$  that is onto to denote such a listing. If the mapping is also one-to-one so that it is a one-to-one correspondence between  $\Omega$  and  $A$ , then we have a "single listing".

The elements of  $U$  may have properties. For example, if  $n \in N$  has the property that  $n = 2m$  for some  $m \in N$ , then  $n$  is said to be **even**. If  $n$  is not even, it is said to be **odd**. Let  $E = \{n \in N: n \text{ is even}\} = \{2, 4, 6, 8, 10, \dots\}$  and  $O = \{n \in N: n \text{ is odd}\} = \{1, 3, 5, 7, 9, \dots\}$ . Since  $\{2m\}_{m=1}^{\infty}$  Then is a (single) listing of  $E$ ,  $E = \{2m: m \in N\}$ . Since this is a single listing, there is a one-to-one correspondence between  $E$  and  $N$ . Hence the cardinality of  $E$  is  $N_0$ .  $E$  and  $O$  are subsets of  $N$  (ie.,  $E, O \in P(N)$ ) and  $E \cup O = N$ . It is possible for an infinite set to be in one-to-one coorespondance with a proper subset. Let  $n \in N$  and  $n \neq 1$ . Then if  $n$  has only the divisors  $n$  and  $1$ , it is said to be **prime**. Thus  $P = \{2, 3, 5, 7, 11, 13, 17, \dots\}$  is the set of prime numbers. If  $x \in R$  can be written as  $x = p/q$  where  $p, q \in Z$  and  $q \neq 0$ , then  $x$  is said to be **rational**. Otherwise,  $x$  is **irrational**.  $Q$  is the set of rational numbers and  $\bar{Q} \subseteq R$  is the set of irrational numbers. If  $x \in R$  is such that  $5x^2 + 2 = 7$ , then  $x$  is said to be a **solution** of the equation  $5x^2 + 2 = 7$ . Let  $S = \{x \in R: 5x^2 + 2 = 7\}$  be the set of all solutions to  $5x^2 + 2 = 7$ ; that is,  $S$  is the **solution set** for  $5x^2 + 2 = 7$ . (Note that solving the equation we obtain the explicit description  $S = \{-1, 1\}$ .) If  $f \in F(R, R)$  is continuous at all point in  $R$ , then  $f$  is said to be continuous. Let  $C(R, R) = \{f \in F(R, R): f \text{ is continuous}\}$ .

Let  $U$  be a space,  $x \in U$  be a variable, and  $C(x)$  be an open statement about a property; that is,  $x$  has the property if  $C(x)$  is true. Then  $S = \{x \in U: C(x)\} \in P(U)$  is the subset of  $U$  containing all elements in  $U$  which have the property. Subsets of  $U$  can have properties also. Let  $a \in U$ . We say that  $A \subseteq U$  contains  $a$  if  $a \in A$ . Then  $S = \{A \subseteq U: A \text{ contains } a\} \in P(P(U))$  is the subset of  $P(U)$  that contains all of the subsets of  $U$  that contain  $a$ . (We usually avoid the word contains as it is used to mean both "has as an element" and "has as a subset".) Recall that  $A \subseteq U$  is finite if, for some  $n \in N$ , there exists a one-to-one correspondence between  $A$  and  $N_n = \{1, 2, \dots, n\}$ . Then  $S = \{A \subseteq U: A \text{ is finite}\} \in P(P(U))$  is the set of all finite subsets of  $U$ . For  $a, b \in R$  with  $a < b$ , we let  $\bar{I} = [a, b] = \{x \in R: a \leq x \leq b\}$  and call  $\bar{I} = [a, b]$  a **finite closed interval** in  $R$ . We let  $Int_c = \{\bar{I} = [a, b]: a < b\} \in P(P(R))$  be the set of all finite closed intervals in  $R$ .

A relation that is reflexive, symmetric and transitive is called an **equivalence relation**. An equivalence relation **partitions** a set into **equivalence classes**. Often we consider all of the elements

in an equivalence class as the same element. An example is fractions. We consider the fractions  $3/4$  and  $6/8$  as the same number even though they are not the same symbol.

Recall that an equivalence relation on  $U$  partitions  $U$  into equivalence classes. If

$\{U_\alpha\}_{\alpha \in \Omega}$  is a single listing of the equivalence classes for the equivalence relation, then

1)  $U_\alpha \cap U_\beta = \emptyset$  if  $\alpha \neq \beta$ . (the sets are mutually exclusive) and

2)  $\bigcup_{\alpha \in \Omega} U_\alpha = U$  (the sets are all inclusive).

Also,  $\tilde{U} = \{U_\alpha : \alpha \in \Omega\}$  is the set of all equivalence classes. If for some  $\alpha \in \Omega$ ,  $f \in U_\alpha$ , we let  $\tilde{U}_f = U_\alpha$ .

If one equivalence class has at least two elements, then  $\{\tilde{U}_f\}_{f \in U}$  is a multiple listing of the

equivalence classes and  $U = \bigcup_{\alpha \in \Omega} U_\alpha = \bigcup_{f \in U} \tilde{U}_f$ . Also,  $\tilde{U} = \{U_\alpha : \alpha \in \Omega\} = \{\tilde{U}_f : f \in U\}$  is the set of all

equivalence classes. If for some  $\alpha \in \Omega$ ,  $f \in U_\alpha$ , we say that  $f$  is a representative element in  $\tilde{U}_f = U_\alpha$ .

Note that if  $f, g \in U$ , then  $\tilde{U}_f = \tilde{U}_g$  iff there is  $\alpha \in \Omega$  such that  $f, g \in U_\alpha$ . An equivalence class has as many names as it has (representative) elements.

Let  $\mathbf{Q}^* = \{(p, q) \in \mathbf{Z}^2, q \neq 0\}$ . Then  $\left\{ \frac{p}{q} \right\}_{(p, q) \in \mathbf{Q}^*}$  is a (multiple) listing of the elements

in  $\mathbf{Q} = \{p/q : (p, q) \in \mathbf{Q}^*\} = \{p/q : p, q \in \mathbf{Z}, q \neq 0\}$ . Let  $\sim$  be the equivalence relation on  $\mathbf{Q}^*$  defined by  $(p, q) \sim (r, s)$  iff  $ps = qr$ . Let  $\tilde{\mathbf{Q}}_{(p, q)}$  be the equivalence class represented by  $(p, q)$ . Then

$\tilde{\mathbf{Q}}^* = \{\tilde{\mathbf{Q}}_{(p, q)}^* : (p, q) \in \mathbf{Q}^*\}$  is the set of all equivalence classes and, with the addition of the appropriate structure, is isomorphic to  $\mathbf{Q}$ .

A geometric example of an equivalence relation is that of vectors interpreted as directed line segments. We consider two directed line segments with the same magnitude and direction to be the same vector. The concept of an equivalence relation defining equivalence classes where all elements in an equivalence class are considered to be the same element appear regularly in more advanced mathematics. For example, piecewise continuous functions that differ only at their points of discontinuity are often considered to be the same function.

Let us assume that **formal logic** (or at least informal logic) and **axiomatic set theory** (or at least naive set theory) have been established. Instead of thinking of the number systems as being discovered (that is that they exist and we simply discover what their properties are) we will think of them as being invented or created in our minds by the axioms we use to describe them. This is the **axiomatic approach to mathematics** originally established by Euclid for geometry about 400 BC. We wish to follow this path not only for all of the number systems, beginning with the **Peano Postulates** and then establishing the **natural numbers**  $\mathbf{N} = \{1,2,3,\dots\}$  followed by **Z**, **Q**, **R**, and **C**, but also for any one of the number systems and indeed for any abstract mathematical structure. Later we will give axioms for the real numbers **R** without first establishing **Q**. We will also give axioms for (i.e., definitions of) the abstract algebraic structures known as a **field** (containing scalars) and a **vector space** (containing vectors). The axiomatic properties for **R** are divided into three categories: The **algebraic** (or **field**) properties, the **order** properties, and the **least upper bound** property. By separating the fundamental properties, we can see which properties are needed to solve particular kinds of problems. We use algebraic (or field) properties to solve algebra problems (e.g., **algebraic equations**), but we need both field properties and order properties to solve **inequalities**. We need the least upper bound property to assure that square roots of all positive real numbers exist. It is important to realize that it is the **structure** on a set that allows us to solve problems such as equations.

Now think of any set that has the algebraic properties of the real numbers, but not necessarily any other properties. Informally, a **field** is a number system where we can add, subtract, multiply, and divide (except by zero of course). In such a set, we can still solve equations by using the algebraic (or field) properties for the set to isolate the unknown variable. Thus we may abstract our notion of solving equations involving real numbers to solving equations using field algebra for the abstract algebraic structure known as a **field**. This abstraction provides economy of thought (and proof) if there is more than one example of a field and there is. **Q**, **R**, and **C** are all fields. Thus if we can establish a problem solving algorithm using field algebra in a field, then this algorithm can be used for solving problems in **Q**, **R**, and **C**. Interestingly, this means that if all of the numbers in an algebraic equation whose solution can be obtained by using field algebra (addition, multiplication, subtraction, and division) are rational, then the solution will be rational. If all of the numbers in an algebraic equation whose solution can be obtained by using field algebra (addition, multiplication, subtraction, and division) are real numbers, then the solution will be a real number (either rational or irrational). If all of the numbers in an algebraic equation whose solution can be obtained by using field algebra (addition, multiplication, subtraction, and division) are complex numbers, then the solution will be a complex number. (It may be real and even rational since these are subsets of **C**.) If all of the numbers in an algebraic equation are rational and computing the square root of a positive number is required, we need the least upper bound property for **R**. If computing the square root of a negative number is required, we need the algebra of **C**.

Thus, to minimize the work required to establish results for solving problems (either for us or more likely for mathematicians), we wish to develop our **problem-solving strategies in abstract settings**. This may be hard at first, but will be economical in the long run. Our abstract



structure will be referred to as an **axiomatic system**. It will consist of axiomatically requiring the existence of one or more sets with axiomatically defined structure, for example, the real number system with all (or just some) of its axiomatically defined properties. Recall that a universal set with structure is called a **space**. Since we are thinking abstractly, the elements in the **abstract space** may or may not have names. The space could be small, but is more likely to have an **infinite** number of elements.

If a set is finite, we can, “at least in theory” identify and name all of the elements if given enough time. However, how do we name all of the elements in an infinite set? For instance, how do we name all of the elements in all of our number systems? Although no one has ever written down names for all of the elements in  $\mathbf{N}$ , we have a naming scheme using the **decimal system** so that, “at least in theory”, we can name all numbers in  $\mathbf{N}$  using the symbols in  $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ . It is important to note that, although we have not written down a name for every number in  $\mathbf{N}$ , given the decimal name for any element in  $\mathbf{N}$ , we can write the name of the next larger number. This is one aspect of the idea (or axiom) of **mathematical induction**. Suppose  $S$  is a countably infinite set. Suppose we choose an element in  $S$  and give it a name. By mathematical induction we agree that having chosen and named  $n$  elements in the set, we can choose and name (via some scheme to insure that all elements have unique names) an element that has not already been named. We accept that all of the elements in the set will eventually get unique names. We say that we have defined them (i.e., we have given them names) **inductively**.

Having named all of the numbers in  $\mathbf{N}$ , we may define and name all of the numbers in  $\mathbf{Z}$  by  $\mathbf{Z} = \{(a,n): a \in \{-,+\} \text{ and } n \in \mathbf{N}\} \cup \{0\}$ . To define and obtain names for the rational numbers we let  $\mathbf{Q}^* = \{(p,q): p,q \in \mathbf{Z}, q \neq 0\}$ . Then  $\mathbf{Q}$  is the set of equivalence classes of  $\mathbf{Q}^*$  determined by the equivalence relation  $\sim$  on  $\mathbf{Q}^*$  defined by  $(p,q) \sim (r,s)$  iff  $ps = qr$ . This means that numbers like  $(3,4)$  (i.e.  $3/4$ ) and  $(6,8)$  (i.e.  $6/8$ ) are considered to be the same number and that they have more than one **fraction name**. It can be shown that all rational numbers not only have fraction names, but also have **decimal names**. These may be infinite, but if they are, there is a repeating pattern. However, the decimal names for real irrational numbers such as  $e$  and  $\pi$  do not have any repeating pattern. Hence, the **decimal system** provides names for all real numbers even though it may take a countably infinite number of symbols from  $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$  to name any given real number. We may define and name a complex number in  $\mathbf{C}$  as  $z = x + yi$  (Euler notation) That is, we may name a complex number by giving two real numbers. Even though we can not physically give the entire name of even one irrational number, since we may compute its decimal name to any specified degree of accuracy, we accept that all real numbers have been (or at least can be) named. Recall that some irrational numbers (such as  $e$  and  $\pi$ ) which are defined by a property have been given names. However, although  $\pi$  has been computed to many, many decimal places, but no one knows its entire decimal name.

If a problem has a unique solution in a set where all of the elements have names, the object of the solution process is to obtain the name of the solution element. If the problem has a large or infinite number of solutions or all elements do not have names, we must define more clearly what we mean by **solving the problem**. This may involve particular kinds of set descriptions for the set of solutions (e.g., solving inequalities) or **approximate solutions** (e.g., finding a rational approximation to an equation whose solution is irrational). We discuss the concept of **mathematical problem solving** in more detail in the next chapter.