Review for Chapter 16

1. Evaluation of double integrals: (x, y coordinates)

$$\int\int_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx, \text{ if } R \text{ is } a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x).$$

$$\int\int_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dydx, \text{ if } R \text{ is } c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y).$$

When you can evaluate the integral by either way, you may want to choose a simpler way.

2. Evaluation of double integrals: (Polar coordinates)

$$\int\int_R f(x,y)dA = \int_a^b \int_{g_1(r)}^{g_2(r)} f(r\cos\theta, r\sin\theta)rd\theta dr, \text{ if } R \text{ is } a \leq r \leq b,\ g_1(r) \leq \theta \leq g_2(r).$$

$$\int\int_R f(x,y)dA = \int_a^b \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta, \text{ if } R \text{ is } a \leq \theta \leq b,\ h_1(\theta) \leq r \leq h_2(\theta).$$

A useful fact: $dA = rdrd\theta = dxdy$.

2. Some applications of double integrals.

2a. Area of region R is $\int\int_R dA$.

2b. The volume beween $z = f(x,y)$ and $z = g(x,y)$ when (x,y) are in R is $\int\int_R (f(x,y) - g(x,y))dA$.

2c. (Applications in Physics) Let $\rho(x,y)$ be the density of lamina whose region is R. Then the mass and the centroid of the lamina $(\overline{x}, \overline{y})$ is

$$\begin{aligned} \text{mass } m = &\ \int\int_R \rho(x,y)dA \\ \overline{x} = &\ \tfrac{1}{m}\int\int_R x\rho(x,y)dA \\ \overline{y} = &\ \tfrac{1}{m}\int\int_R y\rho(x,y)dA \end{aligned}$$

3. Evaluation of triple integrals: (rectangular coordinates) The main idea is the same as the cross section idea. The following gives a way to reduce a triple integral to a double integral.

$$\int\int\int_T f(x,y,z)dV = \int\int_R (\int_{h_2(x,y)}^{h_1(x,y)} f(x,y,z)dz)dA \text{ if } T \text{ is } h_1(x,y) \leq z \leq h_2(x,y),\ (x,y) \text{ in } R.$$

4. Evaluation of triple integrals: (cylindrical coordinates and spherical coordinates) The main relationship among rectangular, cylindrical and spherical coordinates is

$$dV = dxdydz = rdrd\theta dz = \rho^2 \sin\phi d\phi d\theta d\rho.$$

5. Some applications of triple integrals.

5a. The volume of the solid T is $\int\int\int_T dV$.

5b. The mass of a solid T with density $\rho(x, y, z)$ is $\int\int\int_T \rho(x, y, z)dV$.

5c. The centroid of T with density $\rho(x, y, z)$ is $(\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \int\int\int_T x\rho(x, y, z)dV, \overline{y} = \int\int\int_T y\rho(x, y, z)dV, \overline{z} = \int\int\int_T z\rho(x, y, z)dV$$

6. Use double integral to find the surface area. If a curface is given by $\mathbf{r}(u, v) =< x(u, v), y(u, v), z(u, v) >$, where (u, v) is in R, then the area of the curface is

$$\int\int_R |\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}| dA.$$