EXAM 2 - Math 17

NAME:
I.D.:

Instruction: Circle your answers and show all your work **clearly**. Messing around may result in losing credits, since the grader may be forced to pick the worst to grade. Solutions with answer only and without supporting procedures will have little credit. Partial credit will only be given to solutions that contain part of the procedure of a correct solution. **You may leave your answers in fractions or radicals without using calculators to convert them into decimals**.

1. (12 %) Let $\rho^2(\sin^2\theta\cos^2\theta + \cos^2\phi) = 4$ be the equation of a surface. Convert it to rectangular coordinates.

2. (10 %) Evaluate the limit $\lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2+y^2}}$.

3. (12 %) Find an equation of the tangent plane at the point (2,2,1) to the surface

$$z^3 + (x+y)z^2 + (x^2+y^2) = 3.$$

4. (14 %) Find all second-order derivatives of $f(x,y) = \sin(xy^2)$.

5. (12 %) Let $z = x\tan^{-1}(xy)$, where

$$\begin{cases} u = t^2 \\ v = se^t. \end{cases} \quad \text{Find } \frac{\partial z}{\partial t} \text{ and } \frac{\partial z}{\partial s}.$$

6. (12 %) The function $f(x,y) = x^4 + y^4 - 4xy + 1$ has two critical points at $(0,0)$ and $(1,1)$. Classify these critical points.

7. (14 %) Find the highest and the lowest point of the surface given by

$$z = f(x,y) = x^2 + 2xy + 3y^2$$

over a square region with vertices $(-1,0), (-1,1), (1,-1)$ and $(1,1)$.

8. (14 %) Let $f(x,y) = \frac{x}{y}$.

(8A) Find the maximum direcional derivative of f at $(6,-2)$.

(8B) Find the directional derivative of f at the point $(6,-2)$ in the direction $\vec{\mathbf{v}} = -\vec{\mathbf{i}} + 3\vec{\mathbf{j}}$.

(8C) What is the direction in which the function has maximum rate of change at $(6,-2)$, and what is the maximum rate of change at $(6,-2)$?

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1. (12 %) Let $x^2 + y^2 + z^2 = 2\sqrt{x^2 + y^2}$ be the equation of a surface.
   (1A) Convert it to cylindrical coordinates.
   (1B) Convert it to spherical coordinates.

2. (10 %) Show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}$ does not exist.

3. (12 %) Find an equation of the tangent plane at the point (2,2,1) to the surface

$$z^3 + (x + y)z^2 + (x^2 + y^2) = 3.$$

4. (14 %) Find all second-order derivatives of $f(x, y) = \sin(xy^2)$.

5. (12 %) Let $z = xy \ln(u + v)$, where

$$\begin{cases} u = x^2 + y^2 \\ v = x^3 + y^3. \end{cases} \quad \text{Find } \frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}.$$

6. (12 %) The function $f(x, y) = x^3 + 6xy + 3y^2 - 9x$ has two critical points at $(3, -3)$ and $(-1, 1)$. Classify these critical points.

7. (14 %) Find the highest and the lowest point of the surface given by

$$z = f(x, y) = x^2 + 2xy + 3y^2$$

over a triangular region with vertices (0,0), (1,0) and (1,1).

8. (14 %) Let $f(x, y) = xe^{xy}$.
   (8A) Find the maximum direcional derivative of f at $(1, 0)$.
   (8B) Find the directional derivative of f at the point $(1, 0)$ in the direction $\vec{\mathbf{v}} = 3\vec{\mathbf{i}} + 4\vec{\mathbf{j}}$.
   (8C) What is the direction in which the function has maximum rate of change at (1,0), and what is the maximum rate of change at (1,0)?

Review for Exam 2
Functions of Several Variables

1. Cylindrical and Spherical Coordinates

(x, y, z) – rectangular coordinates
(r, θ, z) – cylindrical coordinates
(ρ, ϕ, θ) – spherical coordinates

Conversion Fomulas:
$x = r\cos\theta, y = \sin\theta, r^2 = x^2 + y^2, \tan\theta = \dfrac{y}{x}$ if $x \neq 0$.
$x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi, \rho^2 = x^2 + y^2 + z^2$.

2. Limits of $f(x, y)$ at (a, b).

• $lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$, if $f(x, y)$ is continuous at (a, b).
• If $f(a, b) \to \dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, try to use other coordinates.
• $lim_{(x,y)\to(a,b)} f(x, y)$ exists only if the value of $lim_{(x,y)\to(a,b)} f(x, y)$ remains the same when (x, y) approaches (a, b) along all possible curves.

3. Tangent Plane

• The tangent plane to a surface at (x_0, y_0, z_0) is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

where $\langle a, b, c\rangle = \vec{\mathbf{n}}$ is a normal vector of the tangent plane.

$$\vec{\mathbf{n}} = \begin{cases} \langle f_x(x_0, y_0), f_y(x_0, y_0), -1\rangle & \text{if the surface is } z = f(x, y) \\ \langle F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0)\rangle & \text{if the surface is } F(x, y, z) = 0. \end{cases}$$

• The surface $z = f(x, y)$ has a horizontal tangent plane at (x_0, y_0) if (x_0, y_0) satifies $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

4. Extrema of a function $z = f(x, y)$

• Absolute extrema over a region R:
First find all the candidates (interior critical points of R, critical points on the boundaries of R, cornor points of R, if there are any), then compare their z-values to pick the highest and the lowest values.
• Local maxima and local minima:
First solve $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ to get all the critical points. Then classify each critical point (a, b) by the discreminant

$$\triangle(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b).$$

If $\triangle(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(x, y)$ has a local min at (a, b).
If $\triangle(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(x, y)$ has a local max at (a, b).

If $\triangle(a,b) < 0$, then $f(a,b)$ is neither a local max nor a local min.
If $\triangle(a,b) = 0$, then no conclusion may be made. The $\triangle$ test fails.

5. Partial Derivatives

(A) If $w = f(x,y,z)$, then there are three first order partial dertivatives $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$, and nine second order partial dertivatives.

(B) If $w = f(u,v)$ and $\begin{cases} u = u(t) \\ v = v(t) \end{cases}$ Then $w(t)$ and $w'(t)$ exist and $\frac{dw}{dt} = f_u \frac{du}{dt} + f_v \frac{dv}{dt}$.

(C) If $w = f(u,v)$ and $\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$ Then $w(x,y)$ exists and

$$\frac{\partial w}{\partial x} = f_u \frac{\partial u}{\partial x} + f_v \frac{\partial v}{\partial x}, \qquad \frac{\partial w}{\partial y} = f_u \frac{\partial u}{\partial y} + f_v \frac{\partial v}{\partial y}.$$

(D) If $w = f(x,y,u,v)$ and $\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$ Then $w(x,y)$ exists and

$$\frac{\partial w}{\partial x} = f_u \frac{\partial u}{\partial x} + f_v \frac{\partial v}{\partial x} + f_x, \qquad \frac{\partial w}{\partial y} = f_u \frac{\partial u}{\partial y} + f_v \frac{\partial v}{\partial y} + f_y.$$

6. Directional Derivatives of $f(x,y,z)$

- The **gradiaent vector** $\nabla f = f_x \vec{\mathbf{i}} + f_y \vec{\mathbf{j}} + f_z \vec{\mathbf{k}}$.
- The directional derivative of f on the direction $\vec{\mathbf{u}}$ is $D_{\vec{\mathbf{u}}} f = \nabla f \cdot \vec{\mathbf{u}}$, where $\vec{\mathbf{u}}$ must be a unit vector. $D_{\vec{\mathbf{u}}} f$ is the rate of change of the function in the direction $\vec{\mathbf{u}}$.
- The maximum directional derivative of f at (a,b) is $|\nabla f(a,b)|$.
- ∇f points to the direction in which f has the maximum rate of change.

Solution of Exam 2

1. (12 %) Let $x^2+y^2+z^2=2\sqrt{x^2+y^2}$ be the equation of a surface.
   (1A) Convert it to cylindrical coordinates.
   (1B) Convert it to spherical coordinates.

Solution (1A) $r^2+z^2=2r$, using $x^2+y^2=r^2$.
(1B) Use $\rho^2=x^2+y^2+z^2$ to get

$$\rho^2=2\sqrt{\rho^2\cos^2\phi\cos^2\theta+\rho^2\cos^2\phi\sin^2\theta}=2\rho\sqrt{\cos^2\phi}.$$

Thus the answer is $\rho=2\cos\phi$.

2. (10 %) Show that the limit $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2}{x^2+y^2}$ does not exist.

Solution $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2}{x^2+y^2}=\lim_{(r,\theta)\to(0,\theta)}\frac{r^2\cos^2\theta}{r^2}=\cos^2\theta$. Therefore the answer varies as θ changes. For example, if $\theta=0$, (this means (x,y) goes to (0,0) along the x-axis), then the answer is 1; and if $\theta=\frac{\pi}{4}$, (this means (x,y) goes to (0,0) along the line $y=x$), then the answer is $\frac{\sqrt{2}}{2}$. Since the answer varies as the limiting process takes different curves, the limit does not exists.

3. (12 %) Find an equation of the tangent plane at the point (2,2,1) to the surface $z^3+(x+y)z^2+(x^2+y^2)=3$.

Solution $\vec{\mathbf{n}}=\langle F_x,F_y,F_z\rangle=\langle z^2+2x,z^2+2y,3z^2+2(x+y)z\rangle$, $\vec{\mathbf{n}}(2,2,1)=\langle 5,5,11\rangle$. The equation of the tangent plane is

$$5(x-2)+5(y-2)+11(z-1)=0.$$

4. (14 %) Find all second-order derivatives of $f(x,y)=\sin(xy^2)$.

Solution $f_x=y^2\cos(xy^2)$, $f_y=2xy\cos(xy^2)$. $f_{xx}=-y^4\sin(xy^2)$, $f_{yy}=2x\cos(xy^2)-4x^2y^2\sin(xy^2)$, and $f_{xy}=f_{yx}=2y\cos(xy^2)-2xy^3\sin(xy^2)$.

5. (12 %) Let $z=xy\ln(u+v)$, where

$$\begin{cases} u=x^2+y^2 \\ v=x^3+y^3. \end{cases} \quad \text{Find } \frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}.$$

Solution

$$\frac{\partial z}{\partial x}=f_u\frac{\partial u}{\partial x}+f_v\frac{\partial v}{\partial x}+f_x=\frac{xy}{u+v}(2x)+\frac{xy}{u+v}(3x)+y\ln(u+v),$$

$$\frac{\partial z}{\partial y} = f_u \frac{\partial u}{\partial y} + f_v \frac{\partial v}{\partial y} + f_y = \frac{xy}{u+v}(2y) + \frac{xy}{u+v}(3y)^2 + x\ln(u+v).$$

6. (12 %) The function $f(x,y) = x^3 + 6xy + 3y^2 - 9x$ has two critical points at $(3,-3)$ and $(-1,1)$. Classify these critical points.

Solution $f_x = 3x^2 + 6y - 9$, $f_{xx} = 6x$, $f_{xy} = 6$, $f_y = 6y + 6x$, $f_{yy} = 6$.
$\nabla(3,-3) = 18(6) - 36 > 0$ and $f_{xx}(3,-3) = 18 > 0$, a local min at $(3,-3)$
$\nabla(-1,1) = (-6)(6) - 36 < 0$, neither local max nor local min at $(-1,1)$.

7. (14 %) Find the highest and the lowest point of the surface given by

$$z = f(x,y) = x^2 + 2xy + 3y^2$$

over a triangular region with vertices (0,0), (1,0) and (1,1).

Solution
• Interior points: Solve $f_x = 2x + 2y = 0$ and $f_y = 2x + 6y = 0$ to get $x = 0, y = 0$. (0,0) is a corner point. So there is no critical point inside the triangle.
• Critical Points on the Boundaries:
On the edge $y = x$: $f(x,y) = f(x,x) = x^2 + 2x^2 + 3x^2 = 6x^2, f' = 12x, x = 0. \Longrightarrow$ (0,0).
On the edge $x = 1$: $f(x,y) = f(1,y) = 1 + 2y + 3y^2, f' = 2 + 6y = 0, y = \frac{-1}{3}. \Longrightarrow$ Not in domain. Discarded.
On the edge $y = 0$: $f(x,y) = f(x,0) = x^2, f' = 2x, x = 0. \Longrightarrow$ (0,0).
• There are three corner points (0,0), (1,1), (1,0).
• All together there are four candidates. Compare their z-values:
$f(0,0) = 0$ (minimum value)
$f(1,0) = 1$
$f(1,1) = 1 + 2 + 3 = 6$ (maximum value)
$f(1,\frac{1}{3}) = 1 + \frac{2}{3} + \frac{3}{9} = 1$
The highest point is at (1,1) and the lowest point at (0,0).

8. (14 %) Let $f(x,y) = xe^{xy}$.
(8A) Find the maximum direcional derivative of f at $(1,0)$.
(8B) Find the directional derivative of f at the point $(1,0)$ in the direction $\vec{\mathbf{v}} = 3\vec{\mathbf{i}} + 4\vec{\mathbf{j}}$.
(8C) What is the direction in which the function has maximum rate of change, and what is the maximum rate of change?

Solution
(8A) $\nabla f = f_x\vec{\mathbf{i}} + f_y\vec{\mathbf{j}} = (e^{xy} + xye^{xy})\vec{\mathbf{i}} + x^2e^{xy}\vec{\mathbf{j}}$. $\nabla f(1,0) = \vec{\mathbf{i}} + \vec{\mathbf{j}}$, $|\nabla f| = \sqrt{2}$.
(8B) $\vec{\mathbf{u}} = \dfrac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} = \dfrac{1}{5}(3\vec{\mathbf{i}} + 5\vec{\mathbf{j}})$. $D_{\vec{\mathbf{v}}}f = D_{\vec{\mathbf{u}}}f = \nabla f(1,0) \cdot \vec{\mathbf{u}} = \langle 1,1\rangle \cdot \langle \frac{3}{5}, \frac{4}{5}\rangle = \frac{7}{5}$.
(8C) The direction is $\nabla f(1,0) = \dfrac{1}{\sqrt{2}}\langle 1,1\rangle$, the max. rate of change is $|\nabla f(1,0)| = \sqrt{2}$.