

Infinite Limits and Limits at Infinity

(1) **Infinity Limits** In the limiting process of a fraction, if the denominator approaches 0, while the numerator remains or approaches a positive (negative, respectively) constant, the limiting quantity is positive and can be arbitrarily large (negative with arbitrarily large absolute value, respectively), and so the limit **does not exist**. For convenience, we also write $\lim_{x \rightarrow a} f(x) = \infty$ (or $-\infty$, respectively) to indicate the fact that the limiting quantity can be positively arbitrarily large, (negative with arbitrarily large absolute value, respectively). (**Note that ∞ is not a number**).

(2) **Limits at Infinity** In the limiting process, if x increases unboundedly, then we write $x \rightarrow \infty$; if x decreases unboundedly, then we write $x \rightarrow -\infty$.

Facts about limits at infinity:

- (i) If n is a positive real number, then $\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0$.
(ii) If n is a positive real number, then $\lim_{x \rightarrow \infty} x^n = \infty$.

Example 1 Compute the limits $\lim_{x \rightarrow 1^-} \frac{1-2x}{x^2-1}$, $\lim_{x \rightarrow 1^+} \frac{1-2x}{x^2-1}$ and $\lim_{x \rightarrow 1} \frac{1-2x}{x^2-1}$.

Solution: When $x \rightarrow 1$, $1-2x \rightarrow 1-2 = -1 \neq 0$. When $x < 1$ and $x \rightarrow 1$, we have $0 < x^2 < 1$ and so $x^2 - 1 < 0$. Therefore, as $x \rightarrow 1^-$, both $1-2x < 0$ and $x^2 - 1 < 0$, and so $\lim_{x \rightarrow 1^-} \frac{1-2x}{x^2-1} = \infty$.

When $x \rightarrow 1$, $1-2x \rightarrow 1-2 = -1 \neq 0$. When $x > 1$ and $x \rightarrow 1$, we have $x^2 > 1$ and so $x^2 - 1 > 0$. Therefore, as $x \rightarrow 1^+$, $1-2x < 0$ but $x^2 - 1 > 0$, and so $\lim_{x \rightarrow 1^+} \frac{1-2x}{x^2-1} = -\infty$.

Combine these two limits to conclude that $\lim_{x \rightarrow 1} \frac{1-2x}{x^2-1}$ does not exist.

Example 2 Compute the limits $\lim_{x \rightarrow -2^+} \frac{-x}{\sqrt{4-x^2}}$ and $\lim_{x \rightarrow -2^-} \frac{-x}{\sqrt{4-x^2}}$.

Solution: When $x \rightarrow -2$, $-x \rightarrow 2 \neq 0$. When $x > -2$ and $x \rightarrow -2$, we have $\sqrt{4-x^2} \rightarrow 0$, and so $\lim_{x \rightarrow -2^+} \frac{-x}{\sqrt{4-x^2}} = \infty$.

When $x \rightarrow -2$, $-x \rightarrow -2 \neq 0$. When $x < -2$ and $x \rightarrow -2$, we have $\sqrt{4-x^2} \rightarrow 0$, and so $\lim_{x \rightarrow -2^-} \frac{-x}{\sqrt{4-x^2}} = -\infty$.

Example 3 Compute the limits $\lim_{x \rightarrow \infty} \frac{\sin x}{x}$ and $\lim_{x \rightarrow -\infty} \frac{\cos x}{x}$.

Solution: Note that $|\sin x| \leq 1$. Therefore, $|\frac{\sin x}{x}| \leq \frac{1}{|x|}$. As $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$, we also have $\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$.

One can also use Squeeze Law to compute this limit. From $-1 \leq \sin x \leq 1$. we have, for $x > 0$, $-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$. As $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$, by Squeeze Law, we have $\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$. Similarly, $\lim_{x \rightarrow -\infty} \frac{\cos x}{x} = 0$.

Example 3 Compute the limits $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 3x} - x)$.

Solution: Use the identity $(A - B)(A + B) = A^2 - B^2$ to remove the square root before computing the limit:

$$\begin{aligned} \lim_{x \rightarrow \infty} (\sqrt{x^2 + 3x} - x) &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + 3x} - x)(\sqrt{x^2 + 3x} + x)}{\sqrt{x^2 + 3x} + x} = \lim_{x \rightarrow \infty} \frac{(x^2 + 3x) - x^2}{\sqrt{x^2 + 3x} + x} \\ &= \lim_{x \rightarrow \infty} \frac{3x}{\sqrt{x^2 + 3x} + x} = \lim_{x \rightarrow \infty} \frac{3}{\sqrt{1 + \frac{3}{x}} + 1} = \frac{3}{\sqrt{1+0} + 1} = \frac{3}{2}. \end{aligned}$$

Asymptotes

(1) A line $y = L$ is a **horizontal asymptote** of the graph of $y = f(x)$ if

$$\text{either } \lim_{x \rightarrow \infty} f(x) = L \text{ or } \lim_{x \rightarrow -\infty} f(x) = L.$$

(2) A line $x = x_0$ is a **vertical asymptote** of the graph of $y = f(x)$ if x_0 is **not** in the domain of $f(x)$ and if

$$\text{either } \lim_{x \rightarrow x_0^+} |f(x)| = \infty \text{ or } \lim_{x \rightarrow x_0^-} |f(x)| = \infty.$$

(3) A non vertical line $y = ax + b$ is a **slant asymptote** of the graph $y = f(x)$ if

$$\text{either } \lim_{x \rightarrow \infty} [f(x) - (ax + b)] = 0 \text{ or } \lim_{x \rightarrow -\infty} [f(x) - (ax + b)] = 0.$$

Example 4 Determine all horizontal, slant, and vertical asymptotes of $f(x) = \frac{x}{\sqrt{4+x^2}}$.

Solution: First consider the domain of $f(x)$. Since $4+x^2 > 0$ for all reals, the domain of $f(x)$ is $(-\infty, \infty)$. Therefore, $y = f(x)$ has no vertical asymptotes.

To determine if $y = f(x)$ has a horizontal asymptote, compute

$$\lim_{x \rightarrow \infty} \frac{x}{\sqrt{4+x^2}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{\frac{4}{x^2} + 1}} = \frac{1}{\sqrt{\lim_{x \rightarrow \infty} \frac{4}{x^2} + 1}} = \frac{1}{1} = 1.$$

Therefore, $y = 1$ is a horizontal asymptote and $y = f(x)$ does not have a slant asymptote.

Example 5 Determine all horizontal, slant, and vertical asymptotes of $f(x) = \frac{x^2}{4-x^2}$.

Solution: First consider the domain of $f(x)$. Since $4-x^2 = 0$ if and only if $x = 2$ and $x = -2$, the domain of $f(x)$ is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. Therefore, $y = f(x)$ has possible vertical asymptotes at $x = -2$ and $x = 2$.

As $\lim_{x \rightarrow -2^-} \frac{x^2}{4-x^2} = -\infty$ and $\lim_{x \rightarrow -2^+} \frac{x^2}{4-x^2} = \infty$, $x = -2$ is a vertical asymptote. As $\lim_{x \rightarrow 2^-} \frac{x^2}{4-x^2} = \infty$ and $\lim_{x \rightarrow 2^+} \frac{x^2}{4-x^2} = -\infty$, $x = 2$ is also a vertical asymptote.

To determine if $y = f(x)$ has a horizontal asymptote, compute

$$\lim_{x \rightarrow \infty} \frac{x^2}{4-x^2} = \lim_{x \rightarrow \infty} \frac{1}{\frac{4}{x^2} - 1} = \frac{1}{\lim_{x \rightarrow \infty} \frac{4}{x^2} - 1} = \frac{1}{-1} = -1.$$

Therefore, $y = -1$ is a horizontal asymptote and $y = f(x)$ does not have a slant asymptote.

Example 6 Determine all horizontal, slant, and vertical asymptotes of $f(x) = \frac{x^3}{4-x^2}$.

Solution: First consider the domain of $f(x)$. Since $4-x^2 = 0$ if and only if $x = 2$ and $x = -2$, the domain of $f(x)$ is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. Therefore, $y = f(x)$ has possible vertical asymptotes at $x = -2$ and $x = 2$.

As in the previous example, both $x = -2$ and $x = 2$ are vertical asymptotes.

To determine if $y = f(x)$ has a horizontal asymptote, compute

$$\lim_{x \rightarrow \infty} \frac{x^3}{4-x^2} = \lim_{x \rightarrow \infty} \frac{x}{\frac{4}{x^2} - 1} = -\infty.$$

Therefore, $y = f(x)$ does not have a horizontal asymptote.

To determine if $y = f(x)$ has a slant asymptote, apply division to get

$$\frac{x^3}{4-x^2} = -x + \frac{4}{4-x^2}, \text{ and } \lim_{x \rightarrow \pm\infty} \frac{4}{4-x^2} = 0.$$

Therefore, $y = x$ is a slant asymptote of $y = f(x)$.