Exponential and Logarithmic Functions

Useful Facts:

(1) **Laws of Exponents**
(i) For integers m and n, $x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$.
(ii) For any real number p, $x^{-p} = \frac{1}{x^p}$.
(iii) For any reals p and q, $(x^p)^q = x^{pq}$.
(iv) For any reals p and q, $x^p \cdot x^q = x^{p+q}$.

(2) The **base** b of an **exponential function** $f(x) = b^x$ must be positive. The **domain** of $f(x) = b^x$ is $(-\infty, \infty)$ with $x^0 = 1$, and its **range** is $(0, \infty)$.
(3) When $b \neq 1$, then inverse function of $f(x) = b^x$ is the **logarithm function** with base b, written $\log_b(x)$, defined by

$$y = \log_b(x) \text{ if and only if } x = b^y.$$

The **domain** of $\log_b(x)$ is $(0, \infty)$ with $\log_b(1) = 0$ and $\log_b(b) = 1$, and the **range** of $\log_b(x)$ is $(-\infty, \infty)$.
(4) **Properties of Logarithms** For $b \neq 1$, and for any reals x and y,
(i) $\log_b(xy) = \log_b(x) + \log_b(y)$.
(ii) $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$.
(iii) $\log_b(x^y) = y \log_b(x)$.
(iv) **Change of Bases** For a positive a,

$$\log_b x = \frac{\log_a x}{\log_a b}. \text{ In particular, when } a = e,\ \log_b x = \frac{\ln x}{\ln b}.$$

(5) **Notation:** When $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n = 2.7...$ we also write $\ln(x)$ for $\log_e(x)$; when $b = 10$, we also write $\log(x)$ for $\log_{10}(x)$.
(6) Facts relating the two functions: for $b > 0$ and $b \neq 1$,

$$b^{\log_b(x)} = x \text{ and } \log_b(b^x) = x,$$

and for the natural base e:

$$e^{\ln(x)} = x \text{ and } \ln(e^x) = x.$$

Example 1 Convert the exponent $4^{\frac{-2}{3}}$ into fractional or root form.

Solution: $4^{\frac{-2}{3}} = \frac{1}{4^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{4^2}} = \frac{1}{\sqrt[3]{16}}$.

Example 2 Convert the expression $\frac{1}{2\sqrt{x}}$ into exponent form.

Solution: $\frac{1}{2\sqrt{x}} = (2\sqrt{x})^{-1} = 2^{-1} \cdot x^{\frac{-1}{2}}$.

Example 3 Find the value of $\frac{\sqrt{8}}{2^{\frac{1}{2}}}$

Solution: Note that $\sqrt{8} = \sqrt{2^3} = 2^{\frac{3}{2}}$ and so apply the laws of exponent to get

$$\frac{\sqrt{8}}{2^{\frac{1}{2}}} = \frac{2^{\frac{3}{2}}}{2^{\frac{1}{2}}} = 2^{\frac{3}{2}} \cdot 2^{-\frac{1}{2}} = 2^{\frac{3}{2}-\frac{1}{2}} = 2^1 = 2.$$

Example 4 Solve the equation $2e^{-2x} = 1$ for x.

Solution: Rewrite the equation as $e^{-2x} = \frac{1}{2}$. Apply natural log both sides get

$$-2x = \ln(e^{-2x}) = \ln(\frac{1}{2}) = \ln(1) - \ln(2) = -\ln(2).$$

Therefore, $x = \frac{\ln 2}{2}$.

Example 5 Solve the equation $2\ln(4x) - 1 = 6$ for x.

Solution: Rewrite the equation as $\ln(4x) = \frac{7}{2}$. Apply Facts (6) to get

$$4x = e^{\ln(4x)} = e^{\frac{7}{2}} = \sqrt{e^7},$$

and so $x = \frac{\sqrt{e^7}}{4}$.

Example 6 Use definition of logarithm to determine $\log_3(9), \log_4(64)$ and $\log_3(\frac{1}{27})$.

Solution: Note that $9 = 3^2$, $64 = 4^3$ and $27 = 3^3$. Apply Properties of Logarithms to get

$$\begin{aligned}\log_3(9) &= \log_3(3^2) = 2\log_3(3) = 2\\ \log_4(64) &= \log_4(4^3) = 3\log_4(4) = 3\\ \log_3(\frac{1}{27}) &= \log_3(1) - \log_3(3^3) = -3\log_3(3) = -3.\end{aligned}$$

Example 7 Rewrite the expression $\ln(\frac{3}{4}) + 4\ln(2)$ as a single logarithm.

Solution: Apply Properties of Logarithms to write

$$\ln(\frac{3}{4}) + 4\ln(2) = \ln(\frac{3}{4}) + \ln(2^4) = \ln\left(\frac{3 \cdot 2^4}{4}\right) = \ln(3 \cdot 4) = \ln 12.$$

Example 8 The Richter magnitude M of an earthquake is defined in terms of the energy E in joules released by the earthquake, with $\log_{10}(E) = 4.4 + 1.5M$. Find the energy for earthquakes with magnitudes (a) 4, (b) 5, and (c) 6. For each increase in M of one, by what factor does E change?

Solution: The energy $E = 10^{\log_{10}(E)} = 10^{4.4+1.5M}$. Therefore,

$$\begin{aligned} &\text{when } M = 4 \qquad E = 10^{4.4+1.5\cdot 4} = 10^{10.4} \\ &\text{when } M = 5 \qquad E = 10^{4.4+1.5\cdot 5} = 10^{11.9} \\ &\text{when } M = 4 \qquad E = 10^{4.4+1.5\cdot 6} = 10^{13.4}. \end{aligned}$$

Apply the properties of logarithms to compute

$$\frac{10^{4.4+1.5(M+1)}}{10^{4.4+1.5M}} = \frac{(10^{4.4+1.5M}) \cdot 10^{1.5}}{10^{4.4+1.5M}} = 10^{1.5}.$$

Therefore, for each increase in M of one, by E changes by a factor of $10^{1.5}$.