The Mean Value Theorems

Rolle's Theorem Suppose that $f(x)$ is continuous on $[a, b]$ and is differentiable in (a, b). If $f(a) = f(b)$, then there exists a point c in $(a.b)$ such that $f'(c) = 0$.
The Mean Value Theorem Suppose that $f(x)$ is continuous on $[a, b]$ and is differentiable in (a, b). Then there exists a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example 1 Given $f(x) = \frac{1}{x}$ on the interval $[-1, 1]$, show that there is no value of c in the interval $[-1, 1]$ that makes the conclusion of the Mean Value Theorem true.

Solution: Note that $f'(x) = \frac{-1}{x^2}$. Here $a = -1$ and $b = 1$. The Mean Value Theorem would assure that there is a number c such that

$$-\frac{1}{c^2} = f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{2}{2} = 1.$$

Since a negative number cannot be equal to a positive number, we cannot find such a number c.

Why this would happen? This is because that $f(x)$ has a discontinuity $x = 0$ inside the interval $[-1, 1]$ and so the hypothesis of the Mean Value Theorem is not satisfied.

Example 2 Given $f(x) = x^2+1$ on the interval $[-2, 2]$, check the hypotheses of Rolle's Theorem and the Mean Value Theorem, and find a value c that makes the appropriate conclusion true.

Solution: As $f'(x) = 2x$, $f(x)$ is continuous on$[-2, 2]$ and differentiable on $(-2, 2)$. Note that $f(-2) = 4 + 1 = f(2)$, and so the hypothesis of Rolle's Theorem is satisfied.

To find c in $(-2, 2)$ such that $f'(c) = 0$, we solve the equation $2c = f'(c) = 0$ to get $c = 0$.

Example 3 Show that the function $f(x) = \sqrt{x - 1}$ satisfies the hypotheses of The Mean Value Theorem on $[2, 5]$, and find all numbers c in $(2, 5)$ that satisfy the conclusion of that theorem.

Solution: Since $f(x)$ is a composition function of a power function ($f(u) = \sqrt{u}$) and a polynomial ($u = x - 1$), $f(x)$ is continuous in its domain $[1, \infty)$, and differentiable in $(1, \infty)$; and in particular, $f(x)$ is continuous on $[2, 5]$, and differentiable on $(2, 5)$. Thus $f(x)$ satisfies the hypotheses of The Mean Value Theorem on $[2, 5]$.

Compute $f(2) = 1$ and $f(5) = 2$; and $f'(x) = \frac{1}{2\sqrt{x-1}}$. As in this example, $a = 2$ and $b = 5$,

$$\frac{1}{2\sqrt{x - 1}} = f'(x) = \frac{f(b) - f(a)}{b - a} = \frac{2 - 1}{5 - 2} = \frac{1}{3}.$$

Thus $2\sqrt{x - 1} = 3$, and so $4(x-1) = 9$. It follows that $x = \frac{13}{4}$, and so the only point c satisfying the conclusion of that theorem is $c = \frac{13}{4}$.

Use Rolle's Theorem to determine the number of solutions of an equation

Facts: Let $f(x)$ be a function continuous on $[a, b]$ and differentiable in (a, b).
(1) If $f(x) = 0$ has two distinct solutions in $[a, b]$, then $f'(x) = 0$ has one solution in (a, b).
(2) More generally, for an integer $n \geq 2$, if $f(x) = 0$ has n distinct solutions in $[a, b]$, then $f'(x) = 0$ has $n - 1$ distinct solutions in (a, b).

Example 4 Show that $x^3 + 4x - 3 = 0$ has exactly one solution.

Solution: Let $f(x) = x^3 + 4x - 3$. As $f(0) = -3 < 0$ and $f(1) = 2 > 0$, and as $f(x)$ is continuous, by the Intermediate Value Theorem for continuous functions, $f(x) = 0$ must have at least one solution.

As $f'(x) = x^2 + 4 > 0$, $f(x)$ has at most one solution.

Determine if a function is increasing or decreasing

Facts: Let $f(x)$ be a function on an interval I.
(1) If for any pair of points x_1, x_2 in I with $x_1 < x_2$ we always have $f(x_1) > f(x_2)$ (respectively, $f(x_1) < f(x_2)$) then $f(x)$ is **decreasing** (respectively, **increasing**) in the interval I. If for any pair of points x_1, x_2 in I with $x_1 < x_2$ we always have $f(x_1) \leq f(x_2)$ (respectively, $f(x_1) \geq f(x_2)$) then $f(x)$ is **non increasing** (respectively, **non decreasing**) in the interval I.
(2) If $f'(x) > 0$ (respectively, $f'(x) < 0$) for all x in I, then $f(x)$ is increasing (respectively, decreasing) in the interval I.

Example 5 Given $f(x) = x^3 + 5x + 1$, determine if $f(x)$ is increasing, decreasing, or neither.

Solution: As $f'(x) = 3x^2 + 5 > 0$, $f(x)$ is increasing.

Example 6 Given $f(x) = \ln x$, determine if $f(x)$ is increasing, decreasing, or neither.

Solution: The domain of $f(x)$ is $(0, \infty)$. As $f'(x) = \frac{1}{x} > 0$, $f(x)$ is increasing in its domain.