r-hued coloring of sparse graphs

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Abstract

For two positive integers k, r, a (k, r)-coloring (or r-hued k-coloring) of a graph G is a proper k-vertex-coloring such that every vertex v of degree \( d_G(v) \) is adjacent to at least \( \min\{d_G(v), r\} \) distinct colors. The r-hued chromatic number, \( \chi_r(G) \), is the smallest integer k for which G has a (k, r)-coloring. The maximum average degree of G, denoted by mad(G), equals \( \max\{2|E(H)|/|V(H)| : H \text{ is a subgraph of } G \} \).

In this paper, we prove the following results using the well-known discharging method. For a graph G, if mad(G) < \( \frac{12}{5} \), then \( \chi_3(G) \leq 6 \); if mad(G) < \( \frac{2}{3} \), then \( \chi_2(G) \leq 5 \); if G has no \( C_5 \)-components and mad(G) < \( \frac{8}{5} \), then \( \chi_2(G) \leq 4 \).

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1. Introduction

Graphs in this paper are simple and finite. Notations and terminology undefined here are referred to [1]. Let G be a graph with vertex set \( V(G) \) and edge set \( E(G) \). The set of neighbors of a vertex \( v \) is denoted by \( N_G(v) \). We use \( d_G(v) \) and \( \Delta(G) \) to denote the degree of \( v \) and the maximum degree of G, respectively. A vertex of degree k (resp. at least k) is called a k-vertex (resp. \( k^+ \)-vertex). The maximum average degree of G, denoted by mad(G), equals \( \max\{2|E(H)|/|V(H)| : H \text{ is a subgraph of } G \} \).

A graph G is r-regular if each vertex of G has degree r. We use cycles to denote the connected 2-regular graphs and a cycle of length 2k is denoted by \( C_k \).

A path \( P = u_0u_1 \cdots u_k = P_k \) is a k-path of a graph G, if \( u_1, \ldots, u_k \) are 2-vertices and \( u_0, \ u_{k+1} \) are 3-vertices. Vertices \( u_0 \) and \( u_{k+1} \) are called endpoints of P. The collection of l-threads with \( l \geq k \) are \( k^+ \)-threads. Two vertices u and v are loosely adjacent if u and v are contained in some k-thread P.

A k-vertex-coloring (or simply a k-coloring) of a graph G is a mapping \( c : V(G) \to S \), where S is a set of k colors. In general, S is taken to be \( \{1, \ldots, k\} \). If a vertex adjacent to u is colored i, then we say that u sees i. Otherwise, we say that u misses i. If \( W \subseteq V(G) \), denote by \( c(W) \) the set of colors received by at least one vertex of W. A k-coloring is proper if no two adjacent vertices receive the same color. As we are only concerned about the proper coloring, we refer to a proper coloring simply as a coloring. A (k, r)-coloring (or r-hued k-coloring) of a graph G is a k-coloring such that each vertex v is adjacent to at least \( \min\{d_G(v), r\} \) distinct colors. The r-hued chromatic number of a graph G, denoted \( \chi_r(G) \), is the minimum k for which G has a (k, r)-coloring. A list assignment L of a graph G is a function that assigns to every vertex v of G a set \( L(v) \) of positive integers.

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Given a list assignment $L$ of $G$, a $(L, r)$-coloring of $G$ is a coloring $c$ such that each vertex $v$ is adjacent to at least $\min \{ d_G(v), r \}$ distinct colors and $c(v) \in L(v)$. The $r$-hued choice number of a graph $G$ is the minimum $k$ such that $G$ has a $(L, r)$-coloring where $|L(v)| = k$ for each vertex $v \in V(G)$, and is denoted by $ch_i(G)$.

The concept of $(k, r)$-colorings was introduced by Lai et al. [5], and an upper bound of $\chi_2$ was first studied in the same paper. In [6], Song et al. showed that, for $K_4$-minor free graphs, $\chi_2(G) \leq r + 3$ if $2 \leq r \leq 3$ and $\chi_2(G) \leq \lfloor 3r/2 \rfloor + 1$ if $r \geq 4$. Song et al. [7] proved that $\chi_r(G) \leq r + 5$ if $G$ is a planar graph of girth at least 6. For any planar graph $G$, $\chi_2(G) \leq 5$ was proved by Chen et al. [2], and they conjectured that with the exception of $C_5$, $\chi_2(G) \leq 4$ for all planar graphs. Kim, et al. [3] verified this conjecture in 2013.

Motivated by above results, we use a discharging method and give upper bounds on the 2-hued and 3-hued chromatic numbers for graphs with different maximum average degree constraints in this paper.

**Theorem 1.1.** If $G$ is a graph with $\text{mad}(G) < \frac{12}{5}$, then $\chi_3(G) \leq 6$.

In fact, we prove a slightly stronger result that $ch_3(G) \leq 6$ for graphs with $\text{mad}(G) < \frac{12}{5}$. See the remark at the end of Section 2.1.

**Theorem 1.2.** If $G$ is a graph with $\text{mad}(G) < \frac{7}{2}$, then $\chi_3(G) \leq 5$.

**Remark.**

1. The bound of $\text{mad}(G) < \frac{7}{2}$ is sharp since $G_0$ as shown in Fig. 1 satisfies that $\text{mad}(G_0) = \frac{7}{2}$ but $\chi_3(G_0) = 6$.

2. The bound $\chi_3(G) \leq 5$ is the best possible bound for which there are infinitely many graphs satisfying $\text{mad}(G) < \frac{7}{2}$ and $\chi_3(G) = 5$. The following are two special cases and the construction of more such graphs.
   (a) $C_5$ and a graph obtained from two edge-disjoint $C_5$ joining at exactly one vertex.
   (b) In general, we define a family of connected graphs
   $\mathcal{F} = \{ G : G$ contains a bridge $e$ such that $G - \{ e \} has a $C_5$ component$\}.$

We claim that each member of $\mathcal{F}$ has 3-hued chromatic number at least 5. Assume $G \in \mathcal{F}$ has an edge $xy$ such that $G - \{ uv \}$ has a $C_5 = wxyzw$ as a component. For any 3-hued coloring $c$ of $G$, $|\{ c(x), c(w), c(v), c(u) \}| = 4$ and $|\{ c(y), c(z) \} \cap \{ c(x), c(w), c(v) \}| = 0$. Hence, $|c(C_5)| = 5$ and $\chi_3(G) \geq 5$. Combined with Theorem 1.2, each graph $G$ of $\mathcal{F}$ with $\text{mad}(G) < \frac{7}{2}$ has $\chi_3(G) = 5$ and we have infinitely many of such graphs in $\mathcal{F}$.

In [4], Kim and Park submitted a proof that a graph $G$ with $\text{mad}(G) < \frac{8}{3}$ satisfies $\chi_2(G) \leq 4$. Observe that $\chi_2(C_5) = 5$ while $\text{mad}(C_5) = 2 < \frac{8}{3}$, which reveals a gap in their results. In this paper, we also fix the proof in [4] and prove the following.

**Theorem 1.3.** Let $G$ be a graph with no $C_5$-components. If $\text{mad}(G) < \frac{8}{3}$, then $\chi_2(G) \leq 4$.

**Remark.** In [4], Kim and Park showed that the bound of $\text{mad}(G) < \frac{8}{3}$ is sharp. Let $G$ be the graph obtained by subdividing every edge of $K_5$ once. It is easy to verify that $\text{mad}(G) = \frac{8}{3}$ but $\chi_2(G) = 5$.

### 2. 3-hued colorings

**Lemma 2.1.** Let $k$ be an integer where $k \geq 4$ and $m \geq 2$ be a real number. If a graph $G$ is a graph with minimum number of vertices such that $\chi_3(G) \geq k + 1$ and $\text{mad}(G) \leq m$, then $G$ is connected and has no 1-vertex.

**Proof.** If $G$ has two or more components, then each of the components of $G$ has a $(3, k)$-coloring and so does $G$, a contradiction to the choice of $G$.

Suppose that $G$ has a vertex $u$ with $d_G(u) = 1$ and $uv \in E(G)$. Denote $G' = G - \{ u \}$. Then $\text{mad}(G') \leq m$ and thus $G'$ has a $(3, k)$-coloring $c$ since $|V(G')| < |V(G)|$. If $v$ sees three colors in $G'$, we have $k - 1 \geq 3$ available options to color $u$. If $v$ sees two or fewer colors, then there are at least $k - 3 \geq 1$ available options to color $u$. In both cases, we can extend the coloring $c$ to $u$, a contradiction to the choice of $G$. ■
Lemma 2.2. Let $G$ be a graph with $\Delta \leq 2$, then $\chi_3(G) \leq 5$.

Proof. Since the maximum degree of $G$ is at most 2, $G$ is a union of vertex-disjoint cycles and paths. It is easy to see that each path has a 3-hued coloring with three colors and each cycle has a 3-hued coloring with at most five colors. Thus $\chi_3(G) \leq 5$. ■

2.1. Proof of Theorem 1.1

Let $G$ be a counterexample to Theorem 1.1 with $|V(G)|$ minimized.

Claim 2.1. $G$ has no two adjacent 2-vertices.

Proof. Suppose that $G$ has two adjacent 2-vertices $x$ and $y$. Note that $G$ is connected by Lemma 2.1 and $\Delta(G) \geq 3$ by Lemma 2.2. We can choose $x$ and $y$ with the property that $x$ is adjacent to a 3\textsuperscript{rd}-vertex $u$. Let $v$ be the other neighbor of $y$ and denote $G' = G - \{x, y\}$. Therefore, $G'$ has 3-hued 6-coloring $c$ since $|V(G')| < |V(G)|$ and $\text{mad}(G') \leq \text{mad}(G)$. Let us extend the coloring $c$ to $x$ first. If $d_c(u) \geq 4$, then $|c(N_G(u))| \geq 3$ nd thus only $c(u)$ and $c(v)$ are the forbidden colors for $x$. If $d_c(u) = 3$, then $|c(N_G(u))| \leq 2$, thus $c(N_G(u)) \cup \{c(u), c(v)\}$ is the set of forbidden colors for $x$. Thus we first extend $c$ to $x$. In the resulting coloring, $y$ has at most five forbidden colors, $\{c(u), c(x), c(v)\} \cup \{c(N_G(v))\}$ when $d_c(u) = 3$ or at most three forbidden color $\{c(u), c(v), c(x)\}$ if $d_c(u) \neq 3$. Hence, we can further extend $c$ to $y$ and the resulting coloring will contradict the assumption that $G$ is a counterexample. ■

Initial Charge: $M(x) = d(x) - 12/5$ for each vertex $x$ in $G$. Since $\text{mad}(G) < 12/5$, we have $\sum_{x \in V(G)} M(x) < 0$. It follows from Lemma 2.1 and Claim 2.1 that, $G$ has no 1-vertices and each 2-vertex is adjacent to at most 2-vertices. Note that each k-vertex where $k \geq 3$ is adjacent to at most k-2-vertices. Hence, we can redistribute the charge of the vertices of $G$ as follows.

Discharging Rule: Each 2-vertex receives $1/5$ from each neighbor.

Denote this new charge by $M'(x)$. Hence, $\sum_{x \in V(G)} M'(x) = \sum_{x \in V(G)} M(x) < 0$.

1. For each 2-vertex $u$, $M'(u) = 2 - 12/5 + 2 \times 1/5 = 0$.
2. For each k-vertex $v$ where $k \geq 3$, $M'(v) = k - 12/5 - k \times 1/5 = (4k - 12)/5 \geq 0$.

Therefore, $M'(x) \geq 0$ for each $x \in V(G)$ and $0 > \sum_{x \in V(G)} M(x) = \sum_{x \in V(G)} M'(v) \geq 0$, a contradiction. This completes the proof of Theorem 1.1. ■

Remark. Note that in Claim 2.1, the choice of available colors for $x$ and $y$ do not depend on the set of colors. Therefore, the above result could be generalized to $\text{ch}_3(G) \leq 6$ for a graph $G$ with $\text{mad}(G) < 12/5$. That is, for every list assignment of size six, there is a 3-hued 6-coloring of $G$ such that each vertex is assigned with a color from its list.

2.2. Proof of Theorem 1.2

Let $G$ be a counterexample to Theorem 1.2 with $|V(G)|$ minimized.

Claim 2.2. $G$ has no 3\textsuperscript{rd}-threads.

Proof. Suppose that $G$ has a 3\textsuperscript{rd}-thread $u_0u_1 \cdots u_{k-1}u_k$ where $k \geq 4$. Let $G' = G - \{u_1, u_2, u_3\}$. Then $G'$ has a 3-hued 5-coloring $c$ since $|V(G')| < |V(G)|$ and $\text{mad}(G') \leq \text{mad}(G)$. Let us extend the coloring $c$ to $u_1$ first. Observe that $u_1$ has at most three forbidden colors. Therefore we have at least two available options to color $u_1$. In the resulting coloring, $u_3$ has at most four forbidden colors and then we can further extend $c$ to $u_3$. After that, $u_2$ has at most four forbidden colors $\{c(u_0), c(u_1), c(u_3), c(u_4)\}$. In the last step, we extend the coloring $c$ to $u_2$ to obtain a 3-hued 5-coloring of $G$, a contradiction to the choice of $G$. ■

Claim 2.3. If $P = uvw$ is a 2-thread of $G$, then $d_c(u) = d_c(v) = 3$.

Proof. Suppose that $P = uvw$ be a 2-thread of $G$ in which either $d_c(u) \geq 4$ or $d_c(v) \geq 4$. Without loss of generality, assume $d_c(u) \geq 4$. Let $G' = G - \{x, y\}$. So $G'$ has a 3-hued 5-coloring $c$ by the minimality of $G$. Let us color $y$ first. The worst case is that $y$ has degree three in $G$ and then $y$ would have at most four forbidden colors $\{c(u), c(v)\} \cup c(N_G(v))$. Thus we can always extend the coloring $c$ to $y$. In the resulting coloring, $u$ has already seen three colors in $c$, so $x$ has at most three forbidden colors. Hence, we can further extend the coloring $c$ to $x$, a contradiction to the choice of $G$. ■

Claim 2.4. Let $P = uvw$ be a 2-thread of $G$ and $G' = G - \{x, y\}$. If $c$ is a 3-hued 5-coloring of $G'$, then we can always extend $c$ to $G$ except when $c(N_G(u)) = c(N_G(v))$ and $c(u) \neq c(v)$.

Proof. Suppose that $c$ is a 3-hued 5-coloring of $G'$ such that either $c(N(u)) \neq c(N(v))$ or $c(u) = c(v)$. Let us color $x$ first. By Claim 2.3, $d_c(u) = d_c(v) = 2$. Thus $x$ has at most 4 forbidden colors $c(N_G(u)) \cup \{c(u), c(v)\}$ and we can color $x$ with one of the available options. In the resulting coloring, the set of forbidden colors of $y$ is $c(N_G(v)) \cup \{c(u), c(x), c(v)\}$. 
If \( c(u) = c(v) \), then \( |c(N_G(v)) \cup \{c(u), c(x), c(v)\}| \leq 4 \). If \( c(N_G(u)) \neq c(N_G(v)) \), then we can recolor \( x \) such that \( c(x) \in c(N_G(v)) - c(N_G(u)) \), and therefore \( |c(N_G(v)) \cup \{c(u), c(x), c(v)\}| \leq 4 \). In both cases, we can extend the coloring \( c \) to \( y \), a contradiction to the choice of \( G \).

**Claim 2.5. No 3-vertex is loosely adjacent to five or more 2-vertices.**

**Proof.** Let \( u \) be a 3-vertex of \( G \) such that \( u \) is loosely adjacent to at least five 2-vertices. Since \( G \) has no \( 3^+ \)-vertices by Claim 2.2, \( u \) is a common endpoint of either three 2-edges or two 2-edges and 1-thread (see Fig. 2). Hence, \( d_G(x_1) = d_G(x_2) = d_G(y_1) = d_G(y_2) = d_G(z) = 2 \). By Claim 2.3, \( d_G(x_3) = d_G(y_3) = 3 \). Let \( G' = G - \{u, x_1, x_2, y_1, y_2\} \). Then \( G' \) has a 3-hued 5-coloring \( c \) by the minimality of \( G \).

If \( c(z) \notin c(N_G(y_3)) \), then we can extend the coloring \( c \) to \( u \) first since \( u \) has at most two forbidden colors. In the resulting coloring, \( x_2 \) has at most four forbidden colors, \( \{c(u), c(x_3)\} \cup c(N_G(x_3)) \). Thus we can extend the coloring to \( x_2 \) with one of the available options. Then \( x_1 \) will have at most four forbidden colors \( \{c(z), c(u), c(x_2), c(x_3)\} \), and we can further extend the coloring to \( x_1 \). After that, \( c(N_G(u)) \neq c(N_G(y_3)) \) since \( c(z) \notin c(N_G(y_3)) \). By Claim 2.4, we can extend the coloring to \( \{y_1, y_2\} \), a contradiction to the choice of \( G \). If \( c(z) \notin c(N_G(x_3)) \), we can extend the coloring to \( G \) by symmetry. Hence, we can assume that \( c(z) \in c(N_G(x_3)) \cap c(N_G(y_3)) \). Then \( \{c(x_3), c(z)\} \cup c(N_G(x_3)) = \{c(x_3), c(z)\} \cup c(N_G(y_3)) \).

We first extend the coloring \( c \) to \( x_1 \) by coloring \( x_1 \) with a color not in \( \{c(x_3), c(z)\} \cup c(N_G(x_3)) \), then color \( x_2 \) with a color not in \( \{c(x_1), c(x_3)\} \cup c(N_G(x_3)) \) and then further extend the coloring to \( u \) by coloring it with a color not in \( \{c(x_1), c(x_2), c(z), c(u)\} \). Thus the resulting coloring is a 3-hued 5-coloring of \( G - \{y_1, y_2\} \) and it satisfies \( c(N_G(u)) \neq c(N_G(y_3)) \) since \( c(x_1) \notin c(N_G(y_3)) \). By Claim 2.4, we can finally extend the coloring to \( \{y_1, y_2\} \), a contradiction to the choice of \( G \).

**Initial Charge:** \( M(x) = d(x) - 7/3 \) for each vertex \( x \) in \( G \).

Since \( \text{mad}(G) < 7/3 \), \( \sum_{x \in V(G)} M(x) < 0 \). \( G \) has no 1-vertices by Lemma 2.1. Claim 2.5 says that each 3-vertex is loosely adjacent to at most four 2-vertices. By Claims 2.2 and 2.3, each \( k \)-vertex where \( k \geq 4 \) can only be the endpoint of 1-thread and therefore is loosely adjacent to at most \( k \)-2-vertices. Now we can redistribute the charge as follows.

**Discharging Rule:** Each 2-vertex \( u \) receives \( 1/6 \) from each endpoint of the thread containing \( u \). Denote the new charge by \( M'(x) \). Hence, \( \sum_{x \in E(G)} M'(x) = \sum_{x \in E(G)} M(x) < 0 \).

1. For each 2-vertex \( u \), \( M'(u) = M(u) + 2 \times 1/6 = 2 - 7/3 + 1/3 = 2 - 6/3 = 0 \).
2. For each 3-vertex \( v \), \( M'(v) \geq M(v) - 4 \times 1/6 = 3 - 7/3 - 2/3 = 0 \).
3. For each \( k \)-vertex \( w \) with \( k \geq 4 \), \( M'(w) \geq M(w) - k \times 1/6 = (5k - 14)/6 > 0 \).

Hence, \( M'(x) \geq 0 \) for each \( x \in V(G) \). So \( \sum_{x \in V(G)} M(x) = \sum_{x \in V(G)} M'(x) \geq 0 \), a contradiction. We complete the proof of Theorem 1.2.

3. **Proof of Theorem 1.3**

Let \( G \) be a counterexample to Theorem 1.3 with \( |V(G)| + |E(G)| \) minimized. Then \( G \) must be connected. Otherwise, each component of \( G \) (not a \( C_3 \)) has a 2-hued 4-coloring, and so does \( G \). This would contradict the choice of \( G \).

**Claim 3.1.** \( G \) contains no cycle \( C \) as a subgraph such that \( C = uwxyzu \) and \( w, x, y, z \) are 2-vertices of \( G \).

**Proof.** Suppose that \( G \) contains a cycle \( C = uwxyzu \) where \( w, x, y, z \) are 2-vertices. Since \( G \neq C_5 \), \( d_G(u) \geq 3 \). Let \( G' = G - \{w, x, y\} \). Since \( G \) is connected, so is \( G' \). This implies that \( G' \neq C_5 \) since \( d_G(z) = 1 \). Hence, \( G' \) has a 2-hued 4-coloring \( c \) by the minimality of \( G \). Let us extend the coloring by assigning \( c(u) = c(z), c(x) = a, c(y) = b \) where \( a \neq b \) and \( a, b \notin \{c(u), c(z)\} \). It is easy to verify that the resulting coloring is a 2-hued 4-coloring of \( G \). This contradicts the choice of \( G \).

**Claim 3.2.** \( G \) has no 1-vertex.
\textbf{Proof.} Suppose that $G$ has a vertex $u$ with $d_G(u) = 1$ and $uv \in E(G)$. Denote $G' = G - \{u\}$. Then $G'$ is connected and $G' \neq C_5$ for which would contradict Claim 3.1. Therefore, $G'$ has a 2-hued 4-coloring $c$ by the minimality of $G$. Note that $u$ has at least two available colors. Thus we can extend the coloring $c$ to $u$. This contradicts the assumption that $G$ is a counterexample. \hfill \blacksquare

\textbf{Claim 3.3.} $\Delta(G) \geq 3$.

\textbf{Proof.} Suppose $\Delta(G) \leq 2$. Claim 3.2 says that $G$ has no 1-vertex. Since $G$ is connected, $G$ must be a $C_k$ where $k \neq 5$. Note that, except for $C_5$, every cycle can be 2-hued colored with four or fewer colors. This contradicts the choice of $G$. \hfill \blacksquare

\textbf{Claim 3.4.} $G$ has no two adjacent 2-vertices.

\textbf{Proof.} Suppose that $G$ has two adjacent 2-vertices $x$ and $y$. Since $\Delta(G) \geq 3$ by Claim 3.3, we can choose $x$ and $y$ in a way that $x$ is adjacent to a $3^+$-vertex $u$. Let $v$ be the other neighbor of $y$ and denote $G' = G - \{x, y\}$. Now we consider the following two cases.

\textbf{Case 1.} $G' = C_5$.

By Claim 3.1, $u$ and $v$ are distinct vertices in $C_5$. $G$ must be one of the configurations in Fig. 3. The corresponding 2-hued 4-colorings have been labeled in Fig. 3. This contradicts the choice of $G$.

\textbf{Case 2.} $G' \neq C_5$.

If $G'$ is disconnected, then $G'$ has no $C_3$-components by Claim 3.1. If $G'$ is connected, $G' \neq C_5$ by assumption. In both cases, $G'$ has no $C_3$-components. By the minimality of $G$, $G'$ has a 2-hued 4-coloring $c$. Let us color $y$ first. Note that $y$ has at most three forbidden colors and therefore we can extend $c$ to $y$. Note that $u$ has already seen at least two distinct colors in $c$ since $d_G(u) \geq 2$. Hence, $x$ has at most three forbidden colors, $c(u)$, $c(y)$, and $c(v)$, and therefore we can further extend $c$ to $x$. This contradicts the choice of $G$. \hfill \blacksquare

\textbf{Claim 3.5.} Each 3-vertex in $G$ is loosely adjacent to at most two 2-vertices.

\textbf{Proof.} Suppose that $G$ has a 3-vertex $x$ which is loosely adjacent to at least three 2-vertices. By Claim 3.4, $G$ has no $2^+$-threads. Thus $x$ is adjacent to three 2-vertices, say $\{y_1, y_2, y_3\}$, and each $y_i$ is contained in a 1-thread $xy_i v_i$ for each $i = 1, 2, 3$, where $v_1, v_2$, and $v_3$ are all $3^+$-vertices.

We claim that $x$ is not a cut-vertex. Otherwise, assume that $x$ is a cut-vertex. Then at least one of $\{y_1, y_2, y_3\}$ is a cut-vertex. Without loss of generality, let $y_1$ be a cut-vertex of $G$. Then $xy_1$ is a cut-edge. By Claim 3.1 and since one component has minimum degree 1, no components of $G_1 = G - \{xy_1\}$ is a $C_5$. By the minimality of $G$, $G - \{xy_1\}$ has a 2-hued 4-coloring $c$. Note that $c(y_1) \neq c(v_1)$ and $x$ is in a component that does not contain $y_1$ and $v_1$. Thus we may assume $c(x) \notin \{c(y_1), c(v_1)\}$. Observe that in $G_1$, both $x$ and $v_1$ are $2^+$-vertices. Thus $c$ is a 2-hued 4-coloring of $G_1$, a contradiction to the choice of $G$. This proves that $x$ is not a cut-vertex.

Let $G' = G - \{x, y_1, y_2, y_3\}$. Since $G$ is connected and $x$ is not a cut-vertex, $G'$ is also connected. We consider the following two cases.

\textbf{Case 1.} $G' = C_5$.

If $v_1 = v_2 = v_3$, then $G$ will satisfy the configuration in Claim 3.1, a contradiction. So $v_i \neq v_j$ for some $1 \leq i < j \leq 3$. Hence, $G$ must be one of the configurations in Fig. 4. The corresponding 2-hued 4-coloring has been labeled in Fig. 4. This contradicts the choice of $G$.

\textbf{Case 2.} $G' \neq C_5$.

Since $G'$ is connected, $G'$ has no $C_5$ components. Therefore, $G'$ has a 2-hued 4-coloring $c$ by the minimality of $G$. Since there are 4 colors, we can first extend $c$ to $x$ by coloring it with a color not in $\{c(v_1), c(v_2), c(v_3)\}$. Note that each of $v_i$ has degree at least two in $G'$ and thus sees at least two colors. We first color $y_1$ with a color not in $\{c(x), c(v_1)\}$ and then color $y_2$ with a color not in $\{c(x), c(y_1), c(v_2)\}$ and finally color $y_3$ with a color not in $\{c(x), c(v_3)\}$. It is easy to check that the extension of $c$ is a 2-hued 4-coloring of $G$, a contradiction to the choice of $G$. \hfill \blacksquare

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig3.png}
\caption{Configurations when $G' = C_5$ in Claim 3.4.}
\end{figure}
Claim 3.6. Each 3-vertex in \( G \) is loosely adjacent to at most one 2-vertex.

Proof. By Claim 3.5, suppose that \( G \) has a 3-vertex \( x \) such that \( x \) is loosely adjacent to exactly two 2-vertices, say \( y_1 \) and \( y_2 \). Since \( G \) has no 2-threads, \( x \) is adjacent to \( y_1 \) and \( y_2 \), and each \( y_i \) is contained in a 1-thread \( xy_i v_i \) for each \( i = 1, 2 \). Let \( v_3 \) be the third neighbor of \( x \). Thus \( v_1, v_2, v_3 \) are all 3\(^+\)-vertices.

With a similar argument as in Claim 3.5, we can show that \( x \) is not a cut-vertex. Let \( G' = G - \{x, y_1, y_2\} \). Thus \( G' \) is connected. We consider the following two cases.

Case 1. \( G' = C_5 \).

If \( v_1 = v_2 = v_3 \), then \( G \) will satisfy the configuration in Claim 3.1, a contradiction. So \( v_i \neq v_j \) for some \( 1 \leq i < j \leq 3 \). Hence, \( G \) must be one of the configurations in Fig. 5. The corresponding 2-hued 4-coloring has been labeled in Fig. 5. This contradicts to the choice of \( G \).

Case 2. \( G' \neq C_5 \).

Since \( G' \) is connected, \( G' \) has no \( C_5 \) components. Therefore, \( G' \) has a 2-hued 4-coloring \( c \) by the minimality of \( G \). Note that for each \( i = 1, 2, 3 \), \( d_{G'}(v_i) \geq 2 \) and thus \( v_i \) sees at least two colors. We first color \( x \) with a color not in \( \{c(v_1), c(v_2), c(v_3)\} \), then color \( y_1 \) with a color not in \( \{c(v_1), c(x), c(v_3)\} \), and then color \( y_2 \) with a color not in \( \{c(x), c(v_2)\} \). It is easy to see that this is a 2-hued 4-coloring of \( G \), a contradiction to the choice of \( G \). ■

Initial Charge: \( M(x) = d(x) - 8/3 \) for each vertex \( x \) in \( G \).

Since \( \text{mad}(G) < 8/3 \), \( \sum_{x \in V(G)} M(x) < 0 \). By Claim 3.2, \( G \) has no 1-vertices. By Claim 3.4, each 2-vertex is adjacent to two 3\(^+\)-vertices. Claim 3.6 says that each 3-vertex is adjacent to at most one 2-vertex. By Claim 3.4, each \( k \)-vertex with \( k \geq 4 \) is adjacent to at most \( k \) 2-vertices. Now let us redistribute the charge as follows.

Discharging Rule: Each 2-vertex receives \( 1/3 \) from its two neighbors.

Denote the new charge by \( M'(x) \). Hence, \( \sum_{x \in V(G)} M(x) = \sum_{x \in V(G)} M'(x) < 0 \).

(1) For each 2-vertex \( x \), \( M'(x) \geq 2 - 8/3 + 2 \times 1/3 = 0 \).
(2) For each 3-vertex \( y \), \( M'(y) \geq 3 - 8 / 3 - 1 / 3 = 0 \).

(3) For each \( k \)-vertex \( z \) with \( k \geq 4 \), \( M'(z) \geq k - 8 / 3 - k \times 1 / 3 = (2k - 8) / 3 \geq 0 \).

For any \( x \in V(G) \), \( M'(x) \geq 0 \) and therefore \( \sum_{x \in V(G)} M(x) = \sum_{x \in V(G)} M'(x) \geq 0 \), a contradiction. This completes the proof of Theorem 1.3. ■

References