On dynamic coloring for planar graphs and graphs of higher genus

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For integers \( k, r > 0 \), a \((k, r)\)-coloring of a graph \( G \) is a proper coloring on the vertices of \( G \) by \( k \) colors such that every vertex \( v \) of degree \( d(v) \) is adjacent to vertices with at least \( \min\{d(v), r\} \) different colors. The dynamic chromatic number, denoted by \( \chi_d(G) \), is the smallest integer \( k \) for which a graph \( G \) has a \((k, 2)\)-coloring. A list assignment \( L \) of \( G \) is a function that assigns to every vertex \( v \) of \( G \) a set \( L(v) \) of positive integers. For a given list assignment \( L \) of \( G \), an \((L, r)\)-coloring of \( G \) is a proper coloring of the vertices such that every vertex \( v \) of degree \( d(v) \) is adjacent to vertices with at least \( \min\{d(v), r\} \) different colors, \( c(v) \in L(v) \). The dynamic choice number of \( G \), \( \chi_d^c(G) \), is the least integer \( k \) such that every list assignment \( L \) with \( |L(v)| = k, \forall v \in V(G) \), permits an \((L, 2)\)-coloring. It is known that for any graph \( G \), \( \chi_d^c(G) \leq \chi_d(G) \). Using Euler distributions in this paper, we prove the following results, where (2) and (3) are best possible.

1. If \( G \) is planar, then \( \chi_d^c(G) \leq 6 \). Moreover, \( \chi_d^c(G) \leq 5 \) when \( \Delta(G) \leq 4 \).
2. If \( G \) is planar, then \( \chi_d^c(G) \leq 5 \).
3. If \( G \) is a graph with genus \( g(G) \geq 1 \), then \( \chi_d^c(G) \leq \frac{1}{2}(7 + \sqrt{1 + 48g(G)}) \).

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1. Introduction

Graphs in this paper are simple and finite. For undefined terminologies and notations see [5,18]. Thus for a graph \( G \), \( \Delta(G) \), \( \delta(G) \) and \( \chi(G) \) denote the maximum degree, minimum degree and chromatic number of \( G \) respectively. For \( v \in V(G) \), let \( N_G(v) \) denote the set of vertices adjacent to \( v \) in \( G \), and \( d_G(v) = |N_G(v)| \). Vertices in \( N_G(v) \) are neighbors of \( v \). For an integer \( g \geq 0 \), let \( S_g \) be the orientable surface obtained from the sphere by adding \( g \) Möbius strips (cross-caps). Given an embedding of \( G \) on a closed surface, the genus \( g(G) \) of a graph \( G \) is the minimum number \( g \) such that \( G \) can be embedded on the surface \( S_g \). Let \( G \) be a graph, \( k \geq 0 \) be an integer, \( \bar{k} = \{1, 2, \ldots, k\} \), and \( c : V(G) \mapsto \bar{k} \) be a map. For \( S \subseteq V(G) \), define \( c(S) = \{c(u) \mid u \in S\} \). For integers \( k > 0 \) and \( r > 0 \), a \((k, r)\)-coloring of a graph \( G \) is a map \( c : V(G) \mapsto \bar{k} \) satisfying both the following.

(C1) \( c(u) \neq c(v) \), for every edge \( uv \in E(G) \);
(C2) \( |c(N_G(v))| \geq \min\{d_G(v), r\} \), for every \( v \in V(G) \).

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For a fixed integer \( r > 0 \), the \( r \)-hued chromatic number of \( G \), denoted by \( \chi_r(G) \), is the smallest \( k \) such that \( G \) has a \((k, r)\)-coloring. The concept was first introduced in \([13,11]\), where \( \chi_2(G) \) is called the dynamic chromatic number of \( G \). Later in \([10]\), a referee suggested the name of conditional chromatic number of \( G \). Recently, we received several comments on the name of conditional coloring, suggesting that does not reveal the nature of the coloring. Therefore, we decided to use the name \( r \)-hued chromatic number to reflect the use of many colors near a vertex.

By the definition of \( \chi_r(G) \), it follows immediately that \( \chi(G) = \chi_1(G) \), and so \( r \)-hued coloring is a generalization of the classical graph coloring. Let \( G^2 \) be the graph defined as the following, \( V(G^2) = V(G) \), \( E(G^2) = \{ uv \mid d_G(u, v) \leq 2 \} \), then \( \chi_{\Delta(G)}(G) = \chi(G^2) \). For any integers \( i > j > 0 \), any \((k, i)\)-coloring of \( G \) is also a \((k, j)\)-coloring of \( G \), and so

\[
\chi(G) \leq \chi_2(G) \leq \cdots \leq \chi_{r-1}(G) \leq \chi_r(G) \leq \cdots \leq \chi_{\Delta(G)}(G) = \chi_{\Delta(G)+1}(G) = \cdots.
\]

(1)

A list assignment \( L \) of \( G \) is a function that assigns to every vertex \( v \) of \( G \) a set \( L(v) \) of positive integers. An \( L \)-coloring is a proper coloring \( c \) such that \( c(v) \in L(v) \), for every \( v \in V(G) \). Such coloring is also called list coloring. \( G \) is said to be \( k \)-choosable if, for every list assignment \( L \) with \( |L(v)| = k \), for all \( v \in V(G) \), there exists an \( L \)-coloring of \( G \). The choice number (or list chromatic number) \( ch(G) \) of \( G \), is the least integer \( k \) such that \( G \) is \( k \)-choosable.

There is also a similar generalization for the list coloring. For a given list assignment \( L \) of \( G \) and a given positive integer \( r \), an \( r \)-hued list coloring \( c \) of \( G \) is an \( L \)-coloring of \( G \) such that \( |c(N_G(v))| \geq \min|d_G(v), r| \), for every \( v \in V(G) \). We call such coloring an \((L, r)\)-coloring. The \( r \)-hued choice number (or list chromatic number) \( ch_r(G) \) of \( G \), is the least integer \( k \) such that \( G \) admits an \((L, r)\)-coloring, for any list assignment \( L \) with \( |L(v)| = k \), for every \( v \in V(G) \). Similarly, \( ch_1(G) = ch_1(G) \) and \( ch_{\Delta(G)}(G) = ch_2(G) \). As for any integers \( i > j > 0 \), any \((L, i)\)-coloring of \( G \) is also an \((L, j)\)-coloring of \( G \), it follows

\[
ch(G) \leq ch_2(G) \leq \cdots \leq ch_{r-1}(G) \leq ch_r(G) \leq \cdots \leq ch_{\Delta(G)}(G) = ch_{\Delta(G)+1}(G) = \cdots.
\]

(2)

For any positive integers \( k \) and \( r \), let \( L(v) = k \), for every \( v \) of a graph \( G \). Then every \((k, r)\)-coloring of \( G \) is also an \((L, r)\)-coloring of \( G \), and so

\[
\chi_r(G) \leq ch_k(G).
\]

(3)

Some recent results are published for the case \( r = 2 \). In \([11]\), an analogue of Brooks’ Theorem for \( \chi_2 \) is proved. Akbari et al. \([1]\) proved that \( ch_2(G) \leq \Delta(G) + 1 \) if \( G \) has no component isomorphic to \( C_5 \) and if \( \Delta(G) \geq 3 \). Later in \([7]\), Esperet disproved a conjecture \( ch_2(G) = \max\{ch(G), \chi_2(G)\} \) made in \([1]\). In \([2]\), Alishahi obtained that \( \chi_2(G) \leq \chi(G) + 14.06\ln k+1 \), for any \( k \)-regular graph.

The research for general \( r \) is also of interest. In \([10]\), it is shown that for \( r \geq 2 \), \( \chi_r(G) \leq \Delta(G) + r^2 – r + 1 \) if \( \Delta(G) \leq r \). A Moore graph is a regular graph with diameter \( d \) and girth \( 2d + 1 \). Ding et al. \([6]\) proved that \( \chi_r(G) \leq (\Delta(G))^2 + 1 \), where equality holds if and only if \( G \) is a Moore graph. This is also improved in \([12]\) as \( \chi_r(G) \leq \Delta(G) + 1 \).

The \( r \)-hued coloring for graphs \( G \) embedded on surfaces is of particular interest. The famous Four Color Theorem \([3,4,17]\) and the Heawood formula \([9]\) provide complete answers to the case when \( r = 1 \). Heawood \([9]\) proved that if \( G \) is a connected graph with a 2-cell embedding on \( S^g(G) \), then \( \chi(G) \leq \frac{1}{2}(7 + \sqrt{1 + 48g}) \). The main results of this paper are given below.

**Theorem 1.1.** If \( G \) is a planar graph, then the following hold.

(i) If \( \Delta(G) \leq 4 \), then \( ch_2(G) \leq 5 \);

(ii) \( ch_2(G) \leq 6 \);

(iii) \( \chi_2(G) \leq 5 \).

**Theorem 1.2.** If \( G \) is a graph with genus \( g(G) \geq 1 \), then \( ch_2(G) \leq \frac{1}{2}(7 + \sqrt{1 + 48g}) \).

In Section 2, we present some of the mechanisms to be used in the proofs for the main results. Our main tool is the edge-distribution of a plane graph, which allows us to apply induction in our arguments. The proofs for the two main theorems are presented in the last two sections, respectively.

### 2. Preliminaries

A **plane graph** is a planar graph that is embedded in the plane. Let \( G \) be a connected plane graph, and let \( F \) be a face of \( G \). Then the boundary of \( F \) is the boundary of the open set in the usual topological sense, and it contains the vertices and edges that are incident with \( F \). The degree of \( F \) is the number of edges incident with \( F \). We call the face with degree \( k \) a \( k \)-face.

For a given edge \( e = v_1v_2 \) of \( G \), let \( d_1, d_2 \) denote the degrees of the two endpoints \( v_1 \) and \( v_2 \) of \( e \), and \( d_1^*, d_2^* \) denote the degrees of the two faces adjacent at \( e \), respectively. The **edge contribution** of \( e \) is defined to be \( \Phi(e) = \frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_1^*} + \frac{1}{d_2^*} - 1 \).

The next result is known as a Lebesgue’s formulae.

**Lemma 2.1** (P. 55 in \([14]\)). Let \( G \) be a plane graph, then \( \sum_{e\in E(G)} \Phi(e) = 2 \).
Throughout this paper, for an edge e of a plane graph G, we shall represent the edge configuration of e as the 4-tuple \((x_1, x_2, x_3, x_4)\) such that \(x_1 \leq x_2 \leq x_3 \leq x_4\), where \(\{x_1, x_2, x_3, x_4\} = \{d_1, d_2, d_3, d_4\}\) as multisets. For convenience, we use \((x_1, x_2, x_3, S)\) with \(S\) being a set of integers, to mean that in this configuration, \(x_4\) can be any integer in \(S\). If \(S\) is given by an interval (such in Lemma 2.2), then \(S\) is the set of the integers inside the interval.

**Lemma 2.2.** Let \(G\) be a plane graph with \(\delta(G) \geq 3\). Then there must be an edge with its configuration falling into one of the following categories.

(i) \((3, 3, 3, [3, \infty))\);
(ii) \((3, 3, 4, [4, 11])\);
(iii) \((3, 3, 5, [5, 7])\);
(iv) \((3, 4, [4, 5])\);

**Proof.** We may assume that \(G\) is connected. By Lemma 2.1, \(\sum_{e \in E(G)} \Phi(e) = 2 > 0\), and so \(G\) has an edge \(e\) with \(\Phi(e) > 0\).

We denote the configuration of \(e\) by \((x_1, x_2, x_3, x_4)\). Then \(\sum_{i=1}^{4} \frac{1}{x_i} > 1\).

Since \(\delta(G) \geq 3\), we have \(x_i \geq 3\), for each \(i \in \{1, 2, 3, 4\}\). As \(x_1 \leq x_2 \leq x_3 \leq x_4\), \(4 \cdot \frac{1}{x_4} > 1\), and so \(x_1 < 4\). This implies that \(x_1 = 3\). Thus \(\sum_{i=2}^{4} \frac{1}{x_i} > 1 - \frac{1}{3} = \frac{2}{3}\). As \(3 \cdot \frac{1}{5} < \frac{2}{3}\), this \(x_2 < 5\). It follows that \(x_2 = 3\) or \(x_2 = 4\).

If \(x_2 = 3\), then \(\frac{1}{x_1} + \frac{1}{x_4} > \frac{1}{5}\), hence \(x_3 < 6\). It is routine to verify that if \(x_3 = 3\), then \(x_4\) can be any number no less than 3; if \(x_3 = 4\), then \(4 \leq x_4 \leq 11\); and if \(x_3 = 5\), then \(5 \leq x_4 \leq 7\).

If \(x_2 = 4\), then \(\frac{1}{x_1} + \frac{1}{x_4} > \frac{1}{12}\), and so \(x_3 < 5\). Hence \(x_3 = 4\) and \(x_4 \leq 5\). This completes the proof of the lemma.

By Lemma 2.2, the following properties on the local structure of a plane graph can be obtained.

**Lemma 2.3.** Let \(G\) be a plane graph with \(\delta(G) \geq 3\). Then there must be an edge \(e = v_1v_2\) which meets at least one of the following conditions.

(i) \(d(v_1) \leq 4\) and \(e\) lies in the boundary of a 3-face;
(ii) \(d(v_1) = 3\) and \(e\) lies in the boundary of a 4-face;
(iii) \(d(v_1) = d(v_2) = 3\) and \(e\) is the common boundary of a 5-face and another l-face where \(5 \leq l \leq 7\);
(iv) \(d(v_1) = 5, 5 \leq d(v_2) \leq 7\) and \(e\) is the common boundary of two 3-faces.

**Proof.** By Lemma 2.2, \(G\) has an edge \(e = v_1v_2\) satisfying the conclusion of Lemma 2.2. The conclusions of this lemma will follow by analyzing the four cases listed in Lemma 2.2.

**Lemma 2.4.** Let \(G\) be a smallest counterexample to Theorem 1.1. Then \(G\) must be connected and \(\delta(G) \geq 3\).

**Proof.** We argue by contradiction and assume that \(G\) is a counterexample with \(|V(G)|\) minimized.

Then for some list assignment \(\{L(v) : v \in V(G)\}\), \(G\) has no \((L, 2)\)-coloring. Furthermore, for one such list assignment \(L\) and any \(v \in V(G)\), \(|L(v)| = 5\) if (i) does not hold for \(G\); \(|L(v)| = 6\) if (ii) does not hold for \(G\); \(|L(v)| = \{1, 2, 3, 4, 5\}\) if (iii) does not hold for \(G\). By (4), \(G\) must be connected with \(|V(G)|\) \(\geq 6\).

If \(\delta(G) = 1\), then let \(v\) be a vertex of degree 1 in \(G\) and \(w\) be the only neighbor of \(v\). Denote \(G' = G - v\). By (4), \(G'\) has an \((L, 2)\)-coloring \(c\). Extending \(c\) by coloring \(v\) with \(c(v) \in L(v) \setminus c\{w, w'\}\), where \(w'\) is another neighbor of \(w\). Then \(c\) can be extended to an \((L, 2)\)-coloring for \(G\), contrary to (4).

Now suppose that \(\delta(G) \geq 2\) and \(v\) is a vertex of degree 2. Denote the neighbors of \(v\) as \(x, y\). Let \(x', y'\) be neighbors of \(x, y\) in \(G - v\), respectively. By (4), \(G' = G - v + xy\) has an \((L, 2)\)-coloring \(c\) with \(c(x) \neq c(y)\). Extending \(c\) by coloring \(v\) with \(c(v) \in L(v) \setminus c\{(x, y) \cup \{x', y'\}\}\). Then the extended \(c\) is an \((L, 2)\)-coloring of \(G\), contrary to (4). So we must have \(\delta(G) \geq 3\).

3. Proof of Theorem 1.1

Arguing by contradiction, we assume that \(G\) is a counterexample to Theorem 1.1 with \(|V(G)|\) minimized.

Then for some list assignment \(\{L(v) : v \in V(G)\}\), \(G\) has no \((L, 2)\)-coloring. Equivalently, we may assume that for every \(v \in V(G)\),

\(|L(v)| = 5\), \(\text{if (i) does not hold for } G\);  \(|L(v)| = 6\), \(\text{if (ii) does not hold for } G\);  \(|L(v)| = 5\), \(\text{if (iii) does not hold for } G\).
By Lemma 2.4, \( G \) must be connected with \( \delta(G) \geq 3 \). In the arguments below, we start with a plane graph \( G' \) with \( |V(G')| < |V(G)| \). Then by (5), \( G' \) has an \((L, 2)\)-coloring \( c \). To obtain a contradiction, we extend the \((L, 2)\)-coloring \( c \) on \( G' \) to one on \( G \). In the following arguments, for all unmentioned vertices \( w \) in \( G' \), \( c(w) \) will not be changed in the extension. Throughout this section, let \( e = v_1v_2 \) denote an edge satisfying one of (i)–(iv) in Lemma 2.3. By Lemma 2.3, one of the following four cases must occur.

Case 1. \( d(v_1) \leq 4 \) and \( e \) lies in the boundary of a 3-face.

Let \( G' = G - v_1 \). By (5), \( G' \) has an \((L, 2)\)-coloring \( c \). Extending \( c \) by coloring \( v_1 \) with \( c(v_1) \in L(v_1) \setminus c(N(v_1)) \). As \( \delta(G) \geq 2 \), \( v_1 \) has a pair of adjacent vertices in the 3-face, and so the neighborhood of every vertex has at least 2 different colors. Hence \( c \) is an \((L, 2)\)-coloring of \( G' \), contrary to (5).

Case 2. \( d(v_1) = 3 \) and \( e \) lies in the boundary of a 4-face.

Let \( F_1 = v_1v_2x_1x_2 \) denote the boundary of this 4-face. Let \( G' = G - v_1 + x_2v_2 \). By (5), \( G' \) has an \((L, 2)\)-coloring \( c \). Extending \( c \) by coloring \( v_1 \) with \( c(v_1) \in L(v_1) \setminus c(N(v_1)) \). As \( c \) is an \((L, 2)\)-coloring of \( G' \), \( c(x_2) \neq c(v_2) \). The choice of \( c(v_1) \) makes \( c \) satisfy both \((C1)\) and \((C2)\). And so \( c \) is an \((L, 2)\)-coloring of \( G' \), contrary to (5).

Case 3. \( d(v_1) = d(v_2) = 3 \) and \( e \) is the common boundary of a 5-face and an \( l \)-face where \( 5 \leq l \leq 7 \).

Let \( F_1 \) denote the 5-face, and \( F_2 \) the \( l \)-face. For \( i = 1, 2 \), let \( x_i \) be the neighbor of \( v_i \) on the boundary of \( F_1, v_i \) be the neighbor of \( v_i \) on the boundary of \( F_2 \). Thus \( N(v_1) = \{ x_1, v_1, y_1 \} \) and \( N(v_2) = \{ x_2, y_2, v_1 \} \). Let \( G' = G - v_1 - v_2 \). By (5), \( G' \) has an \((L, 2)\)-coloring \( c \). Extending \( c \) by coloring \( v_1 \) with \( c(v_1) \) from \( L(v_1) \setminus c(N(v_1)) \) and \( c(v_2) \) from \( L(v_2) \setminus c(N(x_2, y_2, x_1, v_1)) \) respectively. As \( c \) is an \((L, 2)\)-coloring of \( G' \), and by the choice of \( c(v_1) \) and \( c(v_2) \), the extended \( c \) satisfies both \((C1)\) and \((C2)\), and so \( c \) is an \((L, 2)\)-coloring of \( G' \), contrary to (5).

Case 4. \( d(v_1) = 5, 5 \leq d(v_2) \leq 7 \) and \( e \) is the common boundary of two 3-faces. (This case is not applicable for Theorem 1.1(iii).)

Suppose that Theorem 1.1(ii) does not hold. By (7), \( |L(v)| = 6 \), for all \( v \in V(G) \). Let \( G' = G - v_1 \). By (5), \( G' \) has an \((L, 2)\)g-coloring \( c \). Since \( d(v_1) = 5 \) in \( G \), \( L(v_1) \setminus c(N_c(v_1)) = \emptyset \). Extending \( c \) by coloring \( v_1 \) with \( c(v_1) \in L(v_1) \setminus c(N(v_1)) \). Since \( e \) lies in a 3-face, \( N_c(v_1) \) contains an edge, and so \( |c(N(v_1))| \geq 2 \). By the definition of \( c(v_1) \) and by the assumption that \( c \) is an \((L, 2)\)-coloring of \( G' \), the extended \( c \) is an \((L, 2)\)-coloring of \( G' \), contrary to (5).

Suppose that Theorem 1.1(iii) does not hold. By (8), \( L(v) = 5 \), for all \( v \in V(G) \). Denote the two faces as \( F_1 = v_1v_2w_1 \) and \( F_2 = v_1v_2w_2 \), respectively. Two subcases are discussed below.

Subcase 4.1. \( w_1w_2 \notin E(G) \).

We obtain \( G' \) from \( G - v_1 \) by identifying \( w_1 \) with \( w_2 \) (denoting the new vertex by \( w \)). Let \( L(w) = 5 \). As \( w_1 \) and \( w_2 \) are in the same face of \( G - v_1 \), \( G' \) is again planar. By (5), \( G' \) has an \((L, 2)\)-coloring \( c \), which can also be viewed as an \((L, 2)\)-coloring of \( G - v_1 \) with \( w_1, w_2 \) receiving the same color. Since \( w_1 \) and \( w_2 \) are identified in \( G' \), \( |c(N_c(v_1))| \leq d_c(v_1) - 1 = 4 \), and so \( L(v_1) \setminus c(N(v_1)) = \emptyset \). Extending \( c \) by coloring \( v_1 \) with \( c(v_1) \in L(v_1) \setminus c(N(v_1)) \). By the definition of \( c(v_1) \) and by the assumption that \( c \) is an \((L, 2)\)-coloring of \( G' \), the extended \( c \) is an \((L, 2)\)-coloring of \( G' \), contrary to (5).

Subcase 4.2. \( w_1w_2 \in E(G) \).

For a plane graph \( G \) with a cycle \( C \), let \( Ext[C] \) (resp. \( Int[C] \)) be the subgraph obtained from \( G \) by deleting all vertices inside (resp. outside) the cycle \( C \). If \( V(Ext[C]) \neq V(C) \) or \( V(Int[C]) \neq V(C) \), then \( C \) is called a separating cycle of \( G \).

Note that the two faces \( F_1 \) and \( F_2 \) must be contained in one of the 3-cycles, \( v_1w_1w_2 \) or \( v_2w_1w_2 \). Without loss of generality, assume that \( C = v_1w_1w_2 \) that contains both \( F_i \) with \( i = 1, 2 \), see Fig. 1. Since both \( d_c(v_i) \geq 5 \) with \( i = 1, 2 \), \( C \) must be a separating cycle of \( G \), and so each of \( Ext[C] \) and \( Int[C] \) has fewer vertices than \( G \).

By (5), each of \( Ext[C] \) and \( Int[C] \) has an \((L, 2)\)-coloring, denoted as \( c_1 \) and \( c_2 \), respectively. Since \( G[v_1, w_1, w_2] \cong K_3 \), we may assume that \( c_1(v_1) = c_2(v_1), c_1(w_1) = c_2(w_1), c_1(w_2) = c_2(w_2) \).

Since \( V(G) = V(Ext[C]) \cup V(Int[C]) \) and \( V(Ext[C]) \cap V(Int[C]) = \{ v_1, w_1, w_2 \} \), and since \( c_1 \) and \( c_2 \) agree on \( \{ v_1, w_1, w_2 \} \), one can construct an \((L, 2)\)-coloring \( c \) of \( G \) by combining \( c_1 \) and \( c_2 \):

\[
c(v) = \begin{cases} 
c_1(v), & \text{if } z \in V(Ext[C]); \\
c_2(v), & \text{if } z \in V(Int[C]). 
\end{cases}
\]
As $c_1$ and $c_2$ are $(L, 2)$-colorings of Ext$[C]$ and Int$[C]$, respectively, and as $G[v_1, w_1, w_2] \cong K_3$, $c$ is an $(L, 2)$-coloring for $G$, contrary to (5). This completes the proof of Theorem 1.1.

As shown in [11], $C_5$ is planar with $\chi_2(C_5) = 5$. It follows by (3) that Theorem 1.1(i) and (iii) are best possible. We conjecture that $C_5$ is the only connected planar graph $G$ with $\chi_2(G) = 5$.

When $r > 2$, the $r$-hued chromatic number $\chi_r(G)$ of a planar graph $G$ may be larger than 5. For example, the wheel $W_5$ with six vertices has $\chi_3(W_5) = 6$, because any pair of vertices of degree 3 that are not adjacent are adjacent to a common vertex of degree 3, and the unique vertex of degree 5 is adjacent to all other vertices. In fact Lai et al. [10] showed that $\chi_r(T) = \min\{r, \Delta(T)\} + 1$ if $T$ is a tree with $|V(T)| \geq 3$. Hence $\chi_5(T) > 5$ if $\Delta(T) \geq 5$.

4. Proof of Theorem 1.2

An embedding of a graph $G$ on an orientable surface (resp. non-orientable surface) $\Sigma$ is minimal if $G$ cannot be embedded on any orientable (resp. non-orientable) surface $\Sigma'$ where $g(\Sigma') < g(\Sigma)$. A graph $G$ is said to have orientable (resp. non-orientable) genus $g$ if $G$ is minimally embedded on a surface with orientable (resp. non-orientable) genus $g$. An embedding of a graph is said to be 2-cell if every face of the embedding is homomorphic to an open unit disk. The Euler characteristic of a graph $G$ is defined as follows.

$$\Phi(G) = \begin{cases} 2 - 2g, & \text{if } G \text{ has the orientable genus } g; \\ 2 - g, & \text{if } G \text{ has the non-orientable genus } g. \end{cases} \quad (9)$$

If $G$ is a connected graph with a 2-cell embedding on a closed surface, then Euler formula indicates that

$$|V(G)| - |E(G)| + |F(G)| = \Phi(G).$$

The following results are needed in our proofs.

**Theorem 4.1** ([19]). If a connected graph $G$ is minimally embedded on an orientable surface, then the embedding is 2-cell.

**Theorem 4.2** ([15]). If $G$ is a connected graph, which is not a tree, then $G$ has a minimal non-orientable embedding which is 2-cell.

Throughout this section, we assume that $G$ is 2-cell embedded on a closed surface. Recall the edge contribution of an edge $e$ is $\Phi(e) = \frac{1}{2} + \frac{1}{g} + \frac{1}{g'} + \frac{1}{\sqrt{2}g'} - 1$. For convenience, let $\Phi'(e) = -\Phi(e)$.

**Lemma 4.3.** If a connected graph $G$ is minimally embedded on a closed surface then

$$\sum_{e \in E(G)} \Phi(e) = \Phi(G).$$

**Proof of Theorem 1.2.** Let $g(G)$ denote the genus of $G$ and $h(G) = \frac{1}{2} \left( 7 + \sqrt{1 + 48g(G)} \right)$. By contradiction, we assume that $G$ is a counterexample to Theorem 1.2$|V(G)|$ minimized. (10)

Then $g(G) \geq 1$, $\chi_2(G) > h(G)$, and $G$ has an assignment $\{L(v) : v \in V(G)\}$ with $|L(v)| = h(G)$, $\forall v \in V(G)$, such that $G$ has no $(L, 2)$-coloring. By (10), $G$ must be connected. We establish each of the following claims. The first claim is an observation following immediately from the definition of $(L, 2)$-colorings.

**Claim 1.** $|V(G)| \geq h(G) + 1$.

**Claim 2.** $\delta(G) \geq h(G) - 2$.

We prove $\delta(G) \geq 3$ first. Let $v$ be a vertex with $d_G(v) = \delta(G)$. If $d_G(v) = 1$, let $N_C(v) = \{w\}, w' \in N_C(w) \setminus \{v\}$ and $G' = G - v$. By (10), $\chi_2(G') \leq h(G')$. By the definition of genus, $g(G') \leq g(G)$, and so $\chi_2(G') \leq h(G') \leq h(G)$. Thus any $(L, 2)$-coloring $c$ of $G'$ can be extended to an $(L, 2)$-coloring of $G$ by coloring $v$ with $c(v) \in L(G) \setminus c([w, w'])$, contrary to (10).

If $d_G(v) = 2$, denote $N_C(v) = \{x, y\}$, and let $x'$ (resp. $y'$) be a neighbor of $x$ (resp. $y$) other than $v$. Let $G' = G - v + x'y$. As $G$ is 2-cell embedded on a surface with $x$ and $y$ on the same face of $G - v$, by the dinition of genus, $g(G') \leq g(G)$. Hence $\chi_2(G') \leq h(G') \leq h(G)$. By (10), $G'$ has an $(L, 2)$-coloring $c$. As $g(G') \geq 1, h(G') > 5$. Hence we can extend $c$ by coloring with $c(x) \in L(G) \setminus c([x, x', y])$. As $c$ is an $(L, 2)$-coloring of $G'$ and by the choice of $c(v), c$ is an $(L, 2)$-coloring of $G$, contrary to (10).

Hence $\delta(G) \geq 3$. We argue by contradiction to prove Claim 2. Assume that $G$ has a vertex $v$ with $d_G(v) \leq h(G) - 3$.

As $\delta(G) \geq 3$, $3x, y \in N_C(v)$ with $x \neq y$. Let $G'' = G - v + xy$. With the same argument above, $g(G'') \leq g(G)$. Hence $\chi_2(G'') \leq h(G'') \leq h(G)$. By (10), $G''$ has an $(L, 2)$-coloring $c$. Let $x', y'$ be a neighbor of $x, y$ in $G - v$, respectively. Extending $c$ by coloring $v$ with $c(v) \in L(G) \setminus c([x, y', x])$. Since $x, y$ are adjacent in $G'$, $c(x) \neq c(y)$. Since $\delta(G) \geq 3, \delta(G') \geq 2$, and so the extended $c$ violates (10). This proves Claim 2.
Claim 3. Let $e = v_1v_2$ be an edge in $G$. Then either $d_1 \geq h(G)$ or $d_2 \geq h(G)$.

We assume otherwise that $d_i = d_c(v_i) \leq h(G) - 1$, $i = 1, 2$. Denote $G' = G - v_1 - v_2$. By (10), $G'$ has an $(L, 2)$-coloring $c$. Denote $N_i = N_c(v_i) \setminus \{v_i\}$, $N_2 = N_c(v_2) \setminus \{v_2\}$. Then $\max(|N_1|, |N_2|) \leq h(G) - 2$. If $\min(|N_1|, |N_2|) \geq 2$, then extend $c$ by coloring $v_1$ with $c(v_1) \in L(v_1) \setminus c(N_1)$ and $v_2$ with $c(v_2) \in L(v_2) \setminus c(N_2 \cup v_1))$. As $c$ is an $(L, 2)$-coloring of $G'$ and by the choices of $c(v_1)$ and $c(v_2)$, $c$ is an $(L, 2)$-coloring of $G$, contrary to (10).

Thus we assume that $|N_2| = 1$. Then pick $v_1' \in N_c(v_1) - \{v_2\}$. Extending $c$ by coloring $v_1$ with $c(v_1) \in L(v_1) \setminus c(N_1 \cup v_2)$ and $v_2$ with $c(v_2) \in L(v_2) \setminus c(N_2 \cup \{v_1, v_1'\})$. As $c$ is an $(L, 2)$-coloring of $G'$ and by the choices of $c(v_1)$ and $c(v_2)$, $c$ is an $(L, 2)$-coloring of $G$, contrary to (10). This proves Claim 3.

Claim 4. Let $e = v_1v_2$ be an edge in $G$. If $3 \notin \{d_1^*, d_2^*\}$, then $d_i \geq h(G)$, $i = 1, 2$.

If not, we assume that $d_1 \leq h(G) - 1$. Let $G' = G - v_1$. Then $g(G') \leq g(G)$, and so by (10), $G'$ has an $(L, 2)$-coloring $c$. Extending $c$ by coloring $v_1$ with $c(v_1) \in L(v_1) \setminus c(N(v_1))$. As $c$ is an $(L, 2)$-coloring of $G'$ and by the choices of $c(v_1)$, $c$ is an $(L, 2)$-coloring of $G$, contrary to (10). This proves Claim 4.

Claim 5. Let $e = v_1v_2$ be an edge in $G$. If $4 \notin \{d_1^*, d_2^*\}$, then $d_i \geq h(G) - 1$, $i = 1, 2$.

If otherwise, we may assume that $d_1^* = 4$ and $d_1 \leq h(G) - 2$. Denote $F = v_1v_2uvwv$ as the 4-face. Let $G' = G - v_1 + wv$. Then by our assumption, $G'$ has an $(L, 2)$-coloring $c$, and so $c(w) \neq c(v_2)$. Extending $c$ by letting $c(v_1) \in L(v_1) \setminus c(N(v_1) \cup \{w\})$, contrary to the choice of $G$. This proves Claim 5.

For notational convenience, we shall denote $h(G)$ and $g(G)$ by $h$ and $g$ respectively throughout the rest of the proof.

Claim 6. Let $e = v_1v_2$ be an edge in $G$. Each of the following holds:

(i) If $3 \in \{d_1^*, d_2^*\}$, then

\[ \Phi'(e) \geq \frac{h - 6}{3h}. \]

(ii) If $3 \notin \{d_1^*, d_2^*\}$, $4 \in \{d_1^*, d_2^*\}$, then

\[ \Phi'(e) \geq \frac{h^2 - 5h + 2}{2h(h - 1)} . \]

(iii) If $d_1^*, d_2^* \geq 5$, then

\[ \Phi'(e) \geq \frac{3h^2 - 16h + 10}{5h(h - 2)}. \]

By Claim 2, $\delta(G) \geq 3$. Thus $d_i \geq 3$, $d_i^* \geq 3$, $i = 1, 2$. If $3 \in \{d_1^*, d_2^*\}$, then by Claim 4, $d_i \geq h$, $i = 1, 2$. Thus $\Phi'(e) = 1 - \frac{1}{d_1} - \frac{1}{d_2} = \frac{h - 6}{3h}$.

If $3 \notin \{d_1^*, d_2^*\}$ and $4 \in \{d_1^*, d_2^*\}$, then by Claim 5, $d_i \geq h - 1$, $i = 1, 2$. By Claim 3, at least one of the $d_i$’s must be at least $h$, and so $\Phi'(e) = 1 - \frac{1}{d_1} - \frac{1}{d_2} = \frac{h^2 - 5h + 2}{2h(h - 1)}$.

If $d_1^*, d_2^* \geq 5$, then by Claim 2, $\delta(G) \geq h - 2$. By Claim 3, at least one of the $d_i$’s must be at least $h(G)$, and so $\Phi'(e) = 1 - \frac{1}{d_1} - \frac{1}{d_2} = \frac{h^2 - 5h + 2}{2h(h - 1)}$. This proves Claim 6.

Since

\[ \frac{h - 6}{3h} \leq \frac{h^2 - 5h + 2}{2h(h - 1)} \leq \frac{3h^2 - 16h + 10}{5h(h - 2)}. \]

(11)

The following claim follows from Claim 6 and (11).

Claim 7. For each $e \in E(G)$,

\[ \Phi'(e) \geq \frac{h - 6}{3h}. \]

Claim 8. $|E(G)| \geq \frac{1}{2}(h + 3)(h - 2)$. 

If $\delta(G) \geq h$, by Claim 1, we have $|V(G)| \geq h + 1$, so $|E(G)| \geq \frac{1}{2}(h + 1)h > \frac{1}{2}(h + 3)(h - 2)$. If $\delta(G) < h$, let $v$ be a vertex of $G$ such that $d(v) = \delta(G)$. Let $u$ be any neighbor of $v$, by Claim 3, $d(u) \geq h$. Thus there exist at least $\delta(G)$ vertices of degree at least $h$, and so $|E(G)| \geq \frac{1}{2}((h + 1)\delta(G) + \delta(G)(h - \delta(G)))$. By Claim 2, $\delta \geq h - 2$. When $\delta(G) = h - 1$, we have that $|E(G)| \geq \frac{1}{2}(h + 2)(h - 1) > \frac{1}{2}(h + 3)(h - 2)$. When $\delta(G) = h - 2$, we have that $|E(G)| \geq \frac{1}{2}(h + 3)(h - 2)$. This proves Claim 8.

By Claim 2, $\delta(G) \geq h - 2 \geq 5$. So $G$ is not a tree. By Theorems 4.1 and 4.2, $G$ has a 2-cell embedding. By Lemma 4.3, $\Phi(G) = \sum_{e \in \mathbb{E}(G)} \Phi(e)$. Since we let $\Phi(e) = -\Phi'(e)$, we have $-\Phi(G) = \sum_{e \in \mathbb{E}(G)} \Phi'(e)$. Now the rest of the proof is divided into 3 cases.

**Case 1.** $\delta(G) \geq h$.

By Claim 1, we have $|V(G)| \geq h + 1$, so $|E(G)| \geq \frac{1}{2}(h + 1)h$.

$$-\Phi(G) = \sum_{e \in \mathbb{E}(G)} \Phi'(e) \geq \frac{1}{2}h(h + 1) \cdot \frac{h - 6}{3h} \cdot \frac{1}{24}(2h)(2h - 10) - 1$$

$$= \frac{1}{24} \left( 7 + \sqrt{1 + 48g} \right) \left( \sqrt{1 + 48g} - 3 \right) - 1 = \frac{1}{24} \left( 48g - 20 + 4\sqrt{1 + 48g} \right) - 1$$

$$= 2g - 2 + \frac{1}{6} \sqrt{1 + 48g} + \frac{1}{6} > 2g - 2. $$

**Case 2.** $\delta(G) = h - 1$.

Let $v$ be the vertex with $d(v) = h - 1$. By Claim 4, every edge $e$ incident to $v$ cannot lie in a 3-face, otherwise we can deduce that $d(v) \geq h$. By Claim 6 and (11), $\Phi'(e) \geq \frac{h^2 - 5h + 2}{2h(h - 1)}$ holds for every edge $e$ incident to $v$.

$$-\Phi(G) = \sum_{e \in \mathbb{E}(G)} \Phi'(e) \geq |E(G)| \cdot \frac{h - 6}{3h} + (h - 1) \left( \frac{h^2 - 5h + 2}{2h(h - 1)} - \frac{h - 6}{3h} \right)$$

$$\geq \frac{1}{2}(h + 3)(h - 2) \cdot \frac{h - 6}{3h} + (h - 1) \left( \frac{h^2 - 5h + 2}{2h(h - 1)} - \frac{h - 6}{3h} \right) = \frac{1}{2} \left( \frac{h^2 - 4h - 13}{3h} \right) + \frac{5}{h}$$

$$= \frac{1}{24} \left( 48g - 6 + 6\sqrt{1 + 48g} \right) - \frac{13}{6} + \frac{5}{h} = 2g - 2 + \frac{1}{12} \left( \frac{3\sqrt{1 + 48g} - 5}{h} \right) > 2g - 2. $$

**Case 3.** $\delta(G) = h - 2$.

Let $v$ be the vertex with $d(v) = h - 2$. By Claims 4 and 5, every edge $e$ incident to $v$ can lie in neither a 3-face nor a 4-face. By Claim 6(iii), $\Phi'(e) \geq \frac{3h^2 - 16h + 10}{5h(h - 2)}$ holds for every edge $e$ incident to $v$.

$$-\Phi(G) = \sum_{e \in \mathbb{E}(G)} \Phi'(e) \geq |E(G)| \cdot \frac{h - 6}{3h} + (h - 2) \left( \frac{3h^2 - 16h + 10}{5h(h - 2)} - \frac{h - 6}{3h} \right)$$

$$\geq \frac{1}{2}(h + 3)(h - 2) \cdot \frac{h - 6}{3h} + (h - 2) \left( \frac{3h^2 - 16h + 10}{5h(h - 2)} - \frac{h - 6}{3h} \right) = \frac{1}{30} \left( 5h^2 - 17h - 76 \right) + \frac{4}{h}$$

$$= \frac{1}{120} \left( 240g + 12 + 36\sqrt{1 + 48g} \right) - \frac{76}{30} + \frac{4}{h} = 2g - 2 + \frac{1}{30} \left( \frac{9\sqrt{1 + 48g} - 13}{h} \right) > 2g - 2. $$

Thus in each case we have $-\Phi(G) > 2g - 2$, contrary to (9). This completes the proof of Theorem 1.2. $\square$

The corollary below follows immediately from Theorem 1.2 and (3).

**Corollary 4.4.** If $G$ is a graph with genus $g(G) \geq 1$, then $\chi_2(G) \leq \frac{1}{2} \left( 7 + \sqrt{1 + 48g(G)} \right)$.

Note that a well-known result by Franklin [8], Ringel [16] and Youngs [19] (see also Theorem 8-8 [18]) states that, for $g(G) \geq 1$, $\chi(G) \leq \frac{1}{2} \left( 7 + \sqrt{1 + 48g(G)} \right)$ is indeed best possible, except for Klein bottle. By formula (1) and (3), $\chi_2(G) \leq \chi_2(G) \leq \chi_2(G)$. So Theorem 1.2 and Corollary 4.4 is also best possible.
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