MAT 378 Exam #2 Take-home Part 2011-04-18

> Guidelines: You may use books and notes. You can't discuss the problems with anyone other than myself. Make sure to read, write and sign the last page.
> Assign each of the questions you solve a point value from $\{3 \cdot 7, 7 \cdot 8, 3 \cdot 9\}$ such that they total exactly 56 points. If you don't assign a point value to a question it will be assigned a value of 7 points.
> These 7 questions will count for a maximum of 56 points.

Name: ______________________________

6. Give a combinatorial argument for the following identity:

$$D_n = n! - \sum_{k=1}^{n} \binom{n}{k} D_{n-k}$$

Solution.
The left hand side is the number of derangements of a set of size n.

The right hand side is the same but considered in the following way. A derangement is a permutation that leaves no elements fixed. Note that $\binom{n}{k} D_{n-k}$ is the number of permutations that fix precisely k elements. So the right hand side is taking all possible permutations and removing those that have k fixed elements for each k 1 through n leaving just those that do not fix any elements (ie. when $k = 0$). (Note that $D_1 = 0$ which matches up with the idea that we cannot fix $n - 1$ elements unless we actually are fixing n elements.) ■

7. How many distinct numbers can be obtained as the product of two or more of the numbers 3,4,4,5,5,6,7,7,7?

No credit will be given for answers that simply enumerate all possibilities.

Solution.
We have 2 choices for the number of 3s (0 or 1); 3 choices for 4s; 3 choices for 5s; 2 choices for 6s; 4 choices for 7s. This makes $2 \cdot 3 \cdot 3 \cdot 2 \cdot 4 = 144$ possibilities. However, we have also counted those numbers that use fewer than two of the numbers from the list. This means we need to remove those that use zero (1 of them) and those that use 1 (9 of them), everything else is acceptable. This yields a final result of $144 - 1 - 5 = 138$ distinct numbers that can be obtained as the product of two or more of the numbers in our list. ■

8. Let $h_n = 8h_{n-1} - 16h_{n-2}$ for $n \geqslant 2$ where $h_0 = -1$ and $h_1 = 0$. Use generating functions to find a (nice) formula for h_n.

Solution.
Let $g(x) = h_0 + h_1x + h_2x^2 + h_3x^3 + \cdots + h_nx^n + \cdots$ be the generating function.

$$\begin{aligned}
g(x) &= h_0 + h_1x + h_2x^2 + h_3x^3 + \cdots + h_nx^n + \cdots \\
-8xg(x) &= \quad -8h_0x - 8h_1x^2 - h_2x^3 - 8h_3x^4 + \cdots - 8h_nx^{n+1} + \cdots \\
16x^2g(x) &= \qquad\qquad 16h_0x^2 + 16h_1x^3 + 16h_2x^4 + 16h_3x^5 + \cdots + 16h_nx^{n+2} + \cdots \\
(1 - 8x + 16x^2)g(x) &= h_0 + (h_1 - 8h_0)x + (h_2 - 8h_1 + 16h_0)x^2 + \cdots + (h_n - 8h_{n-1} + 16h_{n-2})x^n + \cdots \\
(1 - 8x + 16x^2)g(x) &= 8x - 1 \\
g(x) &= \frac{8x-1}{(1-4x)^2} = \frac{A}{1-4x} + \frac{B}{(1-4x)^2}
\end{aligned}$$

So we now have $-4xA + B + A = 8x - 1$, which means $A = -2, B = 1$. So,

$$\begin{aligned}
g(x) &= \frac{-2}{1-4x} + \frac{1}{(1-4x)^2} \\
g(x) &= -2\sum_{k=0}^{\infty} 4^kx^k + \sum_{j=0}^{\infty} \binom{j+1}{j} 4^jx^j \\
g(x) &= \sum_{k=0}^{\infty} (-2 \cdot 4^k + (k+1)4^k)x^k \\
g(x) &= \sum_{k=0}^{\infty} (k-1)4^kx^k
\end{aligned}$$

So we have $h_n = (n-1)4^n$. ∎

9. How many 10 digit phone numbers contain at least one of each odd digit?

Solution.
Let P_i be that the phone number contains no is for $i \in \{1, 3, 5, 7, 9\}$. Thus we have, by inclusion-exclusion,

$$10^{10} - \binom{5}{1}9^{10} + \binom{5}{2}8^{10} - \binom{5}{3}7^{10} + \binom{5}{4}6^{10} - 5^{10}.$$

∎

10. There are 17 students seated in a room with 30 seats. The instructor is not satisfied with the seating arrangement and requests that everyone move to a new seat. How many new seating arrangements are possible?

Solution.
Let P_i be that the ith student **is** in the same seat he started in the new arrangement. Then, by inclusion-exclusion, we have

$$P(30,17)+\sum_{j=1}^{17}(-1)^j\binom{17}{j}P(30-j,17-j)$$

OR

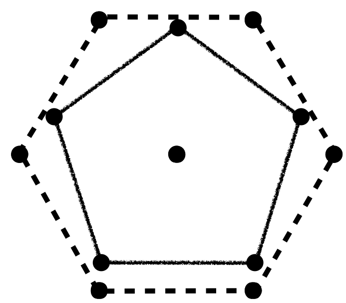
$$\frac{30!}{17!}+\sum_{j=1}^{17}(-1)^j\binom{17}{j}\frac{(30-j)!}{(17-j)!}$$

■

11. Fix a regular hexagon and regular pentagon in the plane centered on the same point precisely as shown in the image below. Let $\mathcal{S}$ denote their vertices along with the center.

    How many equilateral triangles can be formed with at least two vertices in $\mathcal{S}$?

No credit will be given for answers that simply enumerate all triangles.



Solution.
Note that no choice of 2 points from the hexagon and 1 point from the pentagon yield an equilateral triangle, nor does the reverse. Also, two points on the pentagon and the center do not form an equilateral triangle. Nor do 3 points on the pentagon form an equilateral triangle.

This means that the counting can be though of as counting just the hexagon, just the pentagon and then the interaction between the two.

Hexagon: We first notice that we can ignore the center (apparent after the following). Now, we can take any pair of points on the perimeter and count two equilateral triangles for each of these pairings: $\binom{6}{2}$. However, we have counted some things multiple times. If the points are two edges away on the perimeter of the hexagon, the third point of one of the equilateral triangles is also a vertex on the perimeter of the hexagon. This will be counted for each pair on the equilateral triangle (3 pairs) which means we need to remove this twice (leaving it counted once). There are two of this type of triangle, so we have $2 \cdot \binom{6}{2} - 4 = 26$ equilateral triangles for this part.

Pentagon: This is easier since each pair determines precisely two unique equilateral triangles for each choice of points (but we cannot ignore the center this time), and there is no multiple counting going on. Thus there are $2 \cdot \binom{6}{2} = 30$ of these triangles.

Interaction: We choose one point on the hexagon and one point on the pentagon. This determines two unique equilateral triangles for each choice of the two points. This gives $2 \cdot 5 \cdot 6 = 60$ triangles here.

Thus the total number of equilateral triangles is $26 + 30 + 60 = 116$. ■

12. A bakery sells chocolate, cinnamon, plain and crueler doughnuts. Suppose they are attempting to make doughnut eating more exciting by putting the doughnuts in an opaque bag that stacks the doughnuts up on top of each other. To eat them you must take a doughnut out of the bag (off the top of the stack) and eat it, they then repeat until finished. These "Grab Bags" are created so that they contain an even number of chocolate doughnuts, an odd number of plain doughnuts and any number of the other two types.

    How many different grab bags of n doughnuts could be made?

    Hint: Find a generating function for the "Grab Bags". Then find a formula from the generating function.

    Solution.
    The pieces of the generating function are as follows:

$$\begin{aligned} \text{Chocolate: } & 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots & \qquad \text{Chocolate: } & \frac{e^x + e^{-x}}{2} \\ \text{Cinnamon: } & 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots & \text{Cinnamon: } & e^x \\ \text{Plain: } & x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots & \text{Plain: } & \frac{e^x - e^{-x}}{2} \\ \text{Crueler: } & 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots & \text{Crueler: } & e^x \end{aligned}$$

    So,

$$\begin{aligned} g(x) &= \frac{e^x + e^{-x}}{2} \cdot \frac{e^x - e^{-x}}{2} \cdot e^{2x} \\ &= \frac{e^{2x} - e^{-2x}}{4} \cdot e^{2x} \\ &= \frac{e^{4x} - 1}{4} \\ &= -\frac{1}{4} + \frac{e^{4x}}{4} \\ &= -\frac{1}{4} + \sum_{n=0}^{\infty} 4^{n-1} x^n. \end{aligned}$$

    Thus, there are 4^{n-1} different grab bags that could be made for $n > 0$ and 0 for $n = 0$. ■

13. (5 points) ***Bonus:*** Recall that for the in class portion of the exam, you were to clearly mark 5 questions to solve for full credit and the other two would be bonus questions worth 5 points each. On the take-home you were to assign values to all of the problems from the set $\{3 \cdot 7, 7 \cdot 8, 3 \cdot 9\}$ such that the total points was exactly 56 points. Assuming we assigned point values to all questions on the take-home, how many possible ways of completing the two exams and assigning points to each problem on the take home are there?

Solution.
Note that you could assign all seven 8s to the problems on the take home. If you use a 9, you must balance it with a 7 (since the total needs to be exactly 56), so there are a total of $1 + \frac{2!}{1!1!}\binom{7}{2} + \frac{4!}{2!2!}\binom{7}{4} + \frac{6!}{3!3!}\binom{7}{6}$ ways to assign the points on the take-home exam.

$$\binom{7}{5} \cdot \left[1 + \frac{2!}{1!1!}\binom{7}{2} + \frac{4!}{2!2!}\binom{7}{4} + \frac{6!}{3!3!}\binom{7}{6}\right] = 309.$$

■

Once you have finished, please write, in the space provided, “I have neither given nor received help on this test through any means other than using books, notes or asking Ryan Hansen.” and sign your name.