1. How many circular permutations of  $\{3 \cdot a, 4 \cdot b, 2 \cdot c, 2 \cdot d\}$  are there which use all of the elements and in which not all of the same letter appears consecutively?

### Solution.

The total number of circular permutations is (3 + 4 + 2 + 2 - 1)! = 10!. Let  $P_1$  be that the *as* appear consecutively;  $P_2$ , the *bs*;  $P_3$ , the *cs*;  $P_4$ , the *ds*.

$$\begin{aligned} |A_1 \cap A_2| &= \frac{5!}{2!2!} \\ |A_1| &= \frac{8!}{4!2!2!} & |A_1 \cap A_3| &= \frac{7!}{4!2!} & |A_1 \cap A_2 \cap A_3| &= \frac{4!}{2!} \\ |A_2| &= \frac{7!}{3!2!2!} & |A_1 \cap A_4| &= \frac{7!}{4!2!} & |A_1 \cap A_2 \cap A_4| &= \frac{4!}{2!} & |A_1 \cap A_2 \cap A_3 \cap A_4| &= 3! \\ |A_3| &= \frac{9!}{4!3!2!} & |A_2 \cap A_3| &= \frac{6!}{3!2!} & |A_1 \cap A_3 \cap A_4| &= \frac{6!}{4!} \\ |A_4| &= \frac{9!}{4!3!2!} & |A_2 \cap A_4| &= \frac{6!}{3!2!} & |A_2 \cap A_3 \cap A_4| &= \frac{5!}{3!} \\ & |A_3 \cap A_4| &= \frac{8!}{3!4!} \end{aligned}$$

So we have

$$10! - \left(\frac{8!}{4!2!2!} + \frac{7!}{3!2!2!} + \frac{9!}{4!3!2!} + \frac{9!}{4!3!2!}\right) + \left(\frac{5!}{2!2!} + \frac{7!}{4!2!} + \frac{7!}{4!2!} + \frac{6!}{3!2!} + \frac{6!}{3!2!} + \frac{8!}{3!4!}\right) \\ - \left(\frac{4!}{2!} + \frac{4!}{2!} + \frac{6!}{4!} + \frac{5!}{3!}\right) + 3!.$$

2. A bakery sells chocolate, cinnamon and plain doughnuts. At a particular time, they have on hand 6 chocolate, 6 cinnamon and 3 plain doughnuts. How many boxes of 12 doughnuts can be made with the doughnuts available?

#### Solution.

<u>Method 1:</u> Choose the 3 doughnuts to leave out of the 15 to make a dozen. This is the same as the solutions to a + b + c = 3 in non-negative integers. There are  $\binom{5}{3} = 10$  ways.

<u>Method 2</u>: We could use 0, 1, 2 or 3 plain doughnuts. For 0, there is 1 way. For 1, there are 2 ways (replace one cinnamon or one chocolate). For 2, there are 3 ways (replace 2 cinnamon, 2 chocolate or one of each). For 3, there are 4 ways (replace three cinnamon, 3 chocolate, replace 2 cinnamon and 1 chocolate, or replace 2 chocolate and 1 cinnamon). This gives a total of 10 possible boxes of 12 doughnuts that could be made.

<u>Method 3</u>: We can think of balls and barriers (ie. solutions to  $x_1 + x_2 + x_3 = 12$  where  $x_1 \le 6, x_2 \le 6, x_3 \le 3$ ) giving

$$\binom{14}{12} - \left[2 \cdot \binom{7}{5} + \binom{10}{8}\right] + \left[]0 + 2 \cdot \binom{3}{1}\right] - 0.$$

<u>Method 4</u>: We can think of this as the number of integer solutions to a + b + c = 12 where  $3 \le a \le 6, 3 \le b \le 6$  and  $0 \le c \le 3$ . This is the same as the number of integer solutions to

a' + b' + c' = 6 where  $0 \le a' \le 3, 0 \le b' \le 3$  and  $0 \le c' \le 3$ . Let  $P_{a'}, P_{b'}, P_{c'}$  be that a', b' or c' are greater than or equal to 4, respectively. Notice that at most one of these properties can hold, so we have

$$\binom{8}{6} - \binom{3}{1}\binom{4}{2} = 28 - 18 = 10.$$

3. A bakery sells chocolate, cinnamon, plain and crueler doughnuts. Suppose they sell a "Surprise Box" of doughnuts which consists of an even number of chocolate doughnuts, at most one cinnamon doughnut, a multiple of 5 plain doughnuts and at most 4 crueler doughnuts.

How many different "Surprise Boxes" can be made of size n?

*Hint:* Find a generating function for the "Surprise Boxes". Then find a formula from the generating function.

### Solution.

The generating function is given by

$$g(x) = (1 + x^{2} + x^{4} + x^{6} + \dots)(1 + x)(1 + x^{5} + x^{10} + \dots)(1 + x + x^{2} + x^{3} + x^{4})$$
  
=  $\frac{1}{1 - x^{2}} \cdot \frac{1 - x^{2}}{1 - x} \cdot \frac{1}{1 - x^{5}} \cdot \frac{1 - x^{5}}{1 - x}$   
=  $\frac{1}{(1 - x)^{2}}$ 

This is the same as

$$\sum_{k=0}^{\infty} x^k \cdot \sum_{i=0}^{\infty} x^i$$

So we have a coefficient of  $x^n$  whenever k + i = n for k, i non-negative integers. Thus we have the coefficient of  $x^n$  being  $\binom{n+1}{n} = n+1$ . This means there are n+1 ways to make a box of size n.

4. Find a recurrence relation for the number of ternary strings of length n containing at least one pair of consecutive 0's. (ie. 200121000 is OK, but 022010101 is not)

Solution.

Let  $h_n$  be the number of strings of length n containing at least one pair of consecutive 0's. The strings start with 0, 1 or 2. If it starts with 1 or 2, there are  $h_{n-1}$  ways to finish the string. If it starts with a 0, the next number is important. It can start 00, 01 or 02. For 01 and 02, there are  $h_{n-2}$  ways of finishing the string. If it starts 00, then the string be finished arbitrarily so there are  $3^{n-2}$  ways of finishing this type. This yields the recurrence

$$h_n = 2h_{n-1} + 2h_{n-2} + 3^{n-2}$$

5. How many ternary strings of length n contain at least one pair of consecutive symbols that are the same? (This is not the same as problem 4.)

# Solution.

It is easier to count how many ternary strings of length n contain **no** consecutive symbols that are the same. There are  $3^n$  total ternary strings of length n. The string can start with a 0, 1 or 2, but each value after that only has one option eliminated since it cannot be the same as its predecessor. This gives the number of strings of this type as  $3 \cdot 2^{n-1}$ , so the number of strings with at least one pair of consecutive symbols the same is

$$3^n - 3 \cdot 2^{n-1}$$

6. How many ways can four non-attacking rooks be placed on the following board?

		×	
×			
×			×
	×		

Solution.

This problem is easiest by enumeration. Consider a placement of the rooks to be a permutation of  $\{1, 2, 3, 4\}$  where the position indicates the column and the value indicates the row the rook should be placed in. Then the only acceptable setups would be

1234	1324	1342
4321	4231	4132.

7. At the annual Worthington Hooker School<sup>\*</sup> Dance-Till-You-Drop dance-a-thon, 30 couples are "dancing". Of the 30 couples 12 are jock-cheerleader couples. The principal arrives and decides that things are getting a little too steamy. He asks that everyone switch to a new partner. Of course, the jocks again end up with the cheerleaders.

How many new configurations are possible?

## Solution.

Number each jock/cheerleader couple with the same value from  $\{1, 2, ..., 12\}$  and each non-jock/non-cheerleader couple with  $\{1, 2, ..., 18\}$ . Then we can easily see this is just a derangement of the cheerleaders and the non-cheerleaders among their respective partitions, so there are  $D_{12} \cdot D_{18}$  ways to rearrange the couplings.

 $<sup>^{*}</sup>$ Actual high school located in New Haven Conneticut named after former Yale University professor and physician Dr. Worthington Hooker.