1. How many circular permutations of $\{3 \cdot a, 4 \cdot b, 2 \cdot c, 2 \cdot d\}$ are there which use all of the elements and in which not all of the same letter appears consecutively?

## Solution.

The total number of circular permutations is $(3+4+2+2-1)!=10$ !. Let $P_{1}$ be that the $a$ s appear consecutively; $P_{2}$, the $b s ; P_{3}$, the $c \mathrm{~s} ; P_{4}$, the $d \mathrm{~s}$.

$$
\begin{array}{lll} 
& \left|A_{1} \cap A_{2}\right|=\frac{5!}{2!2!} & \\
\left|A_{1}\right|=\frac{8!}{4!2!2!} & \left|A_{1} \cap A_{3}\right|=\frac{7!}{4!2!} & \left|A_{1} \cap A_{2} \cap A_{3}\right|=\frac{4!}{2!} \\
\left|A_{2}\right|=\frac{7!}{3!2!2!} & \left|A_{1} \cap A_{4}\right|=\frac{7!}{4!2!} & \left|A_{1} \cap A_{2} \cap A_{4}\right|=\frac{4!}{2!} \quad\left|A_{1} \cap A_{2} \cap A_{3} \cap A_{4}\right|=3! \\
\left|A_{3}\right|=\frac{9!}{4!3!2!} & \left|A_{2} \cap A_{3}\right|=\frac{6!}{3!2!} & \left|A_{1} \cap A_{3} \cap A_{4}\right|=\frac{6!}{4!} \\
\left|A_{4}\right|=\frac{9!}{4!3!2!} & \left|A_{2} \cap A_{4}\right|=\frac{6!}{3!2!} & \left|A_{2} \cap A_{3} \cap A_{4}\right|=\frac{5!}{3!} \\
& \left|A_{3} \cap A_{4}\right|=\frac{8!}{3!4!} &
\end{array}
$$

So we have

$$
\begin{aligned}
& 10!-\left(\frac{8!}{4!2!2!}+\frac{7!}{3!2!2!}+\frac{9!}{4!3!2!}+\frac{9!}{4!3!2!}\right)+\left(\frac{5!}{2!2!}+\frac{7!}{4!2!}+\frac{7!}{4!2!}+\frac{6!}{3!2!}+\frac{6!}{3!2!}+\frac{8!}{3!4!}\right) \\
& \quad-\left(\frac{4!}{2!}+\frac{4!}{2!}+\frac{6!}{4!}+\frac{5!}{3!}\right)+3!.
\end{aligned}
$$

2. A bakery sells chocolate, cinnamon and plain doughnuts. At a particular time, they have on hand 6 chocolate, 6 cinnamon and 3 plain doughnuts. How many boxes of 12 doughnuts can be made with the doughnuts available?

## Solution.

Method 1: Choose the 3 doughnuts to leave out of the 15 to make a dozen. This is the same as the solutions to $a+b+c=3$ in non-negative integers. There are $\binom{5}{3}=10$ ways.
Method 2: We could use $0,1,2$ or 3 plain doughnuts. For 0 , there is 1 way. For 1 , there are 2 ways (replace one cinnamon or one chocolate). For 2 , there are 3 ways (replace 2 cinnamon, 2 chocolate or one of each). For 3, there are 4 ways (replace three cinnamon, 3 chocolate, replace 2 cinnamon and 1 chocolate, or replace 2 chocolate and 1 cinnamon). This gives a total of 10 possible boxes of 12 doughnuts that could be made.
Method 3: We can think of balls and barriers (ie. solutions to $x_{1}+x_{2}+x_{3}=12$ where $x_{1} \leq 6, x_{2} \leq 6, x_{3} \leq 3$ ) giving

$$
\binom{14}{12}-\left[2 \cdot\binom{7}{5}+\binom{10}{8}\right]+\left[10+2 \cdot\binom{3}{1}\right]-0 .
$$

Method 4: We can think of this as the number of integer solutions to $a+b+c=12$ where $\overline{3 \leq a \leq 6,3} \leq b \leq 6$ and $0 \leq c \leq 3$. This is the same as the number of integer solutions to
$a^{\prime}+b^{\prime}+c^{\prime}=6$ where $0 \leq a^{\prime} \leq 3,0 \leq b^{\prime} \leq 3$ and $0 \leq c^{\prime} \leq 3$. Let $P_{a^{\prime}}, P_{b^{\prime}}, P_{c^{\prime}}$ be that $a^{\prime}, b^{\prime}$ or $c^{\prime}$ are greater than or equal to 4 , respectively. Notice that at most one of these properties can hold, so we have

$$
\binom{8}{6}-\binom{3}{1}\binom{4}{2}=28-18=10 .
$$

3. A bakery sells chocolate, cinnamon, plain and crueler doughnuts. Suppose they sell a "Surprise Box" of doughnuts which consists of an even number of chocolate doughnuts, at most one cinnamon doughnut, a multiple of 5 plain doughnuts and at most 4 crueler doughnuts.
How many different "Surprise Boxes" can be made of size $n$ ?
Hint: Find a generating function for the "Surprise Boxes". Then find a formula from the generating function.

Solution.
The generating function is given by

$$
\begin{aligned}
g(x) & =\left(1+x^{2}+x^{4}+x^{6}+\cdots\right)(1+x)\left(1+x^{5}+x^{10}+\cdots\right)\left(1+x+x^{2}+x^{3}+x^{4}\right) \\
& =\frac{1}{1-x^{2}} \cdot \frac{1-x^{2}}{1-x} \cdot \frac{1}{1-x^{5}} \cdot \frac{1-x^{5}}{1-x} \\
& =\frac{1}{(1-x)^{2}}
\end{aligned}
$$

This is the same as

$$
\sum_{k=0}^{\infty} x^{k} \cdot \sum_{i=0}^{\infty} x^{i}
$$

So we have a coefficient of $x^{n}$ whenever $k+i=n$ for $k, i$ non-negative integers. Thus we have the coefficient of $x^{n}$ being $\binom{n+1}{n}=n+1$. This means there are $n+1$ ways to make a box of size $n$.
4. Find a recurrence relation for the number of ternary strings of length $n$ containing at least one pair of consecutive 0's. (ie. 200121000 is OK, but 022010101 is not)

## Solution.

Let $h_{n}$ be the number of strings of length $n$ containing at least one pair of consecutive 0 's. The strings start with 0,1 or 2 . If it starts with 1 or 2 , there are $h_{n-1}$ ways to finish the string. If it starts with a 0 , the next number is important. It can start 00,01 or 02 . For 01 and 02 , there are $h_{n-2}$ ways of finishing the string. If it starts 00 , then the string be finished arbitrarily so there are $3^{n-2}$ ways of finishing this type. This yields the recurrence

$$
h_{n}=2 h_{n-1}+2 h_{n-2}+3^{n-2}
$$

5. How many ternary strings of length $n$ contain at least one pair of consecutive symbols that are the same? (This is not the same as problem 4.)

## Solution.

It is easier to count how many ternary strings of length $n$ contain $\boldsymbol{n o}$ consecutive symbols that are the same. There are $3^{n}$ total ternary strings of length $n$. The string can start with a 0,1 or 2 , but each value after that only has one option eliminated since it cannot be the same as its predecessor. This gives the number of strings of this type as $3 \cdot 2^{n-1}$, so the number of strings with at least one pair of consecutive symbols the same is

$$
3^{n}-3 \cdot 2^{n-1}
$$

6. How many ways can four non-attacking rooks be placed on the following board?

|  |  | $\times$ |  |
| :---: | :---: | :---: | :---: |
| $\times$ |  |  |  |
| $\times$ |  |  | $\times$ |
|  | $\times$ |  |  |

## Solution.

This problem is easiest by enumeration. Consider a placement of the rooks to be a permutation of $\{1,2,3,4\}$ where the position indicates the column and the value indicates the row the rook should be placed in. Then the only acceptable setups would be

| 1234 | 1324 | 1342 |
| :--- | :--- | :--- |
| 4321 | 4231 | 4132. |

7. At the annual Worthington Hooker School* Dance-Till-You-Drop dance-a-thon, 30 couples are "dancing". Of the 30 couples 12 are jock-cheerleader couples. The principal arrives and decides that things are getting a little too steamy. He asks that everyone switch to a new partner. Of course, the jocks again end up with the cheerleaders.
How many new configurations are possible?

## Solution.

Number each jock/cheerleader couple with the same value from $\{1,2, \ldots, 12\}$ and each non-jock/non-cheerleader couple with $\{1,2, \ldots, 18\}$. Then we can easily see this is just a derangement of the cheerleaders and the non-cheerleaders among their respective partitions, so there are $D_{12} \cdot D_{18}$ ways to rearrange the couplings.

[^0]
[^0]:    *Actual high school located in New Haven Conneticut named after former Yale University professor and physician Dr. Worthington Hooker.

