9. (5 points) Give a combinatorial argument for the following identity:

$$\sum_{j=0}^{k} \binom{n}{j}\binom{n-j}{k-j} = 2^k \binom{n}{k}$$

(Hint: It may be useful to think of coloring some objects.)

No credit will be given for solutions using factorials!

Solution.
The RHS gives the number of ways to choose k houses of n white houses in a neighborhood to paint and then choose whether to paint them red or blue.

The LHS gives the number of ways to choose j houses to paint red and $k-j$ houses to paint blue. Since j takes all values between 0 and k this gives all ways to paint the total of k houses chosen. ■

10. (a) (4 points) Give a combinatorial argument for the following identity:

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$$

No credit will be given for solutions using factorials!

Solution.
The LHS gives the number of ways to choose a committee of k members and select a subcommittee of size m from it's members.

The RHS gives the number of ways to choose the subcommittee first, then fill out the committee with $k-m$ other members for a total of k members on the committee. ■

(b) (3 points) Use the identity in part (a) to show that for $0 < m \leqslant k < n$, $\binom{n}{m}$ and $\binom{n}{k}$ have a nontrivial common factor. In other words, $\gcd\left(\binom{n}{m}, \binom{n}{k}\right) > 1$.
(Hint: Use contradiction.)

Solution. **[Contradiction]**

Method #1: Suppose that $\gcd\left(\binom{n}{m}, \binom{n}{k}\right) = 1$. Notice, $\dfrac{\binom{n}{k}\binom{k}{m}}{\binom{n}{m}} = \binom{n-m}{k-m}$. However, since $\gcd\left(\binom{n}{m}, \binom{n}{k}\right) = 1$, all terms from $\binom{n}{m}$ must cancel with terms from $\binom{k}{m}$ since $\binom{n-m}{k-m}$ is an integer, but $\binom{n}{m} > \binom{k}{m}$. $\lightning$. Thus, $\gcd\left(\binom{n}{m}, \binom{n}{k}\right) > 1$.

Method #2: Suppose that $\gcd\left(\binom{n}{m}, \binom{n}{k}\right) = 1$. This means $\binom{n}{m} \mid \binom{n}{k}\binom{k}{m}$ but $\binom{n}{m} \nmid \binom{n}{k}$. This means $\binom{n}{m} \mid \binom{k}{m}$, however, $\binom{n}{m} > \binom{k}{m}$. $\lightning$. Thus, $\gcd\left(\binom{n}{m}, \binom{n}{k}\right) > 1$. ■

11. Select 55 distinct integers between 1 and 100 (ie. $1 \leqslant x_1 < x_2 < \cdots < x_{55} \leqslant 100$).

(a) (4 points) Show that there are two integers chosen that differ by exactly $10^\dagger$.

Solution.
The method used in (c) works for this as well. Alternatively, create a partition of $\{1, \ldots, 100\}$ as $\Big\{ \{1, 11\}, \{2, 12\}, \ldots, \{90, 100\} \Big\}$. There are 50 sets, but 55 distinct integers must be chosen. PHP gives that one set had both elements chosen from it. These are the two integers chosen that differ by exactly 10. ■

(b) (4 points) Show that there are two integers chosen that differ by exactly $12^\dagger$.

Solution.
The method used in (c) works for this as well. Alternatively, create a partition of $\{1, \ldots, 100\}$ as $\Big\{ \{1, 13\}, \{2, 14\}, \ldots, \{11, 23\}, \{24, 36\}, \{25, 37\}, \ldots, \{83, 95\}, \{96\}, \{97\}, \{98\}, \{99\}, \{100\} \Big\}$. This makes 53 sets, but 55 distinct integers must be chosen. PHP gives that one set had both elements chosen from it. These are the two integers chosen that differ by exactly 12. ■

(c) (4 points) In how many ways can we choose so that no two differ by exactly $15^\ddagger$?

Solution.
We cannot use the alternative methods presented for the previous parts because, the choices in one "box" may affect those in another. This will make counting difficult so we need to find a partition where the parts do not affect each other. Such a partitioning is:

$$\begin{aligned}
S_i &= \{x : 1 \leqslant x \leqslant 100 \text{ and } x \equiv i \bmod 15\} \\
\text{ie. } S_1 &= \{1, 16, 31, 46, 61, 76, 91\} \\
&\vdots \\
S_{10} &= \{10, 25, 40, 55, 70, 85, 100\} \\
S_{11} &= \{11, 26, 41, 56, 71, 86\} \\
&\vdots \\
S_{15} &= \{15, 30, 45, 60, 75, 90\}
\end{aligned}$$

Notice, the sets $S_1, \ldots, S_{10}$ have 7 elements and $S_{11}, \ldots, S_{15}$ have 6. We now have to choose 55 integers. For the sets with 7 elements, we could choose up to 4 non-sequential (within the set) integers making a total of up to 40 integers from those sets. The remaining 5 sets could have up to 3 values chosen from them for up to 15 more integers for a maximum total of 55 integers. This means we have to choose the integers in this manner. The number of ways of choosing 4 non-sequential values from each of $S_1, \ldots, S_{10}$ is one. The number of ways of choosing 3 non-sequential values from S_{11} is 4, so the number of ways of doing this for sets $S_{11}, \ldots, S_{15}$ is 4^5. This means the total number of ways of choosing 55 integers between 1 and 100 such that no pair differs by exactly 15 is $4^5 = 1024$ ways. ■

† This is, in fact, true for 1–7,9, 10, 12, 13, 16, 17, 18, 23–27, 46–54

‡ This can be done for 8,11, 14, 15, 19–22, 28–45, 55 and up

12. (5 points) Let k be a given positive integer. Show that any non-negative integer N can be written uniquely in the form

$$N = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \cdots + \binom{n_1}{1},$$

where $0 \leqslant n_1 < n_2 < \cdots < n_{k-1} < n_k$.

(Hint: Let n be the value such that $\binom{n}{k} \leqslant N < \binom{n+1}{k}$. Then any possible representation has $n_k = n$. [Why?] Then, use induction and that $N - \binom{n}{k} < \binom{n}{k-1}$ [Why is this?] to show the existence and uniqueness of the representation.)

Solution.

Claim 1. *Let n be the value such that $\binom{n}{k} \leqslant N < \binom{n+1}{k}$. Any possible representation of N has $n_k = n$.*

Proof. If $n_k > n$, then $\binom{n_k}{k} > N$ and since none of the other terms are negative, $n_{k-1}, n_{k-2}, \ldots, n_1$ for the representation cannot exist. This means $n_k \leqslant n$.

Suppose $n_k \leqslant n-1$. Then we recall from class when we iterated Pascal's Identity two ways that one of them was:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-2}{k-1} + \binom{n-3}{k-2} + \cdots + \binom{n-k}{1} + 1 \qquad (*)$$

Since $n_k \leqslant n-1$, we have

$$\begin{aligned}\binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \cdots + \binom{n_1}{1} &\leqslant \binom{n-1}{k} + \binom{n-2}{k-1} + \cdots + \binom{n-k}{1} \\ &< \binom{n-1}{k} + \binom{n-2}{k-1} + \cdots + \binom{n-k}{1} + 1 \\ &= \binom{n}{k} \leqslant N.\end{aligned}$$

So we would be at least one short of N. This means $n_k \geqslant n$.

Thus, $n_k = n$ for any such possible representation of N. ■

Claim 2. *If n is the value such that $\binom{n}{k} \leqslant N < \binom{n+1}{k}$, then $N - \binom{n}{k} < \binom{n}{k-1}$.*

Proof. We know $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ so $N - \binom{n}{k} < \binom{n}{k-1}$. ■

Here we begin the inductive proof. For $k = 1$, we have $N = \binom{N}{1}$ as the unique representation for N. We induct on N (and k). Let $k > 1$ and $N \geqslant 0$ be given. Let n be the largest value such that $\binom{n}{k} \leqslant N < \binom{n+1}{k}$. By Claim 1, we know $n_k = n$ for any possible representation of N. We also know that $N - \binom{n_k}{k} \leqslant \binom{n}{k-1} - 1$ by Claim 2. Using equation (*) and the inductive hypothesis, $N - \binom{n}{k}$ can be represented uniquely in the following form where $n_k > n_{k-1} > \cdots > n_1 \geqslant 0$.

$$\begin{aligned} N - \binom{n_k}{k} &= \binom{n_{k-1}}{k-1} + \binom{n_{k-2}}{k-2} + \cdots + \binom{n_1}{1} \\ \implies N &= \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \binom{n_{k-2}}{k-2} + \cdots + \binom{n_1}{1}.\end{aligned}$$

The existence and uniqueness of $n_1, \ldots, n_{k-1}$ is given by the inductive hypothesis and the existence and uniqueness of n_k is given by Claim 1. ■

13. (5 points) Let $n = 2k$ be an even number and $\mathcal{S}$ be a set of n elements. Define a *factor* to be a partition of $\mathcal{S}$ into k sets of size 2. Show that the number of factors is equal to $1 \cdot 3 \cdot 5 \cdots (2k-1)$.

Solution.
Method #1: Pick an element at random and look with what it is paired ($2k - 1$ choices). Pick another element at random and look wit what it is paired ($2k - 3$ choices because the first element, its pair and this element cannot be chosen). Continue until all elements are with a pair completing the factor. This gives the total number of possible factors, $(2k - 1) \cdots 5 \cdot 3 \cdot 1$.

Method #2: Order the elements in a line ($(2k)!$ ways), then pair them up. This will give too many factors so we have to divide by something. The order within each pair doesn't matter so we divide by 2^k. Also, the ordering of the pairs among each other doesn't matter (ie. which pair is first, second, etc.) so we divide by $k!$ (# of pairs factorial) as well. This gives

$$\frac{(2k)!}{2^k \cdot k!} = \frac{(2k)!}{2k \cdot (2k-2) \cdots 2} = (2k-1) \cdots 5 \cdot 3 \cdot 1$$

■