MAT 378 Exam #1 Take-home Part 2011-03-02

GUIDELINES: You may use books and notes. You can't discuss the problems with anyone other than myself. Make sure to read, write and sign the last page. There are 5 questions for a total of 34 points.

Name: ______________________________

9. (5 points) Give a combinatorial argument for the following identity:

$$\sum_{j=0}^{k} \binom{n}{j}\binom{n-j}{k-j} = 2^k \binom{n}{k}$$

(Hint: It may be useful to think of coloring some objects.)

10. (a) (4 points) Give a combinatorial argument for the following identity:

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$$

NO CREDIT WILL BE GIVEN FOR SOLUTIONS USING FACTORIALS!

(b) (3 points) Use the identity in part (a) to show that for $0 < m \leqslant k < n$, $\binom{n}{m}$ and $\binom{n}{k}$ have a nontrivial common factor. In other words, $\gcd\left(\binom{n}{m}, \binom{n}{k}\right) > 1$.
(Hint: Use contradiction.)

11. Select 55 distinct integers between 1 and 100 (ie. $1 \leqslant x_1 < x_2 < \cdots < x_{55} \leqslant 100$).

    (a) (4 points) Show that there are two integers chosen that differ by exactly 10[†].

    (b) (4 points) Show that there are two integers chosen that differ by exactly 12[†].

    (c) (4 points) In how many ways can we choose so that no two differ by exactly 15[‡]?

[†]This is, in fact, true for 1–7,9, 10, 12, 13, 16, 17, 18, 23–27, 46–54

[‡]This can be done for 8,11, 14, 15, 19–22, 28–45, 55 and up

12. (5 points) Let k be a given positive integer. Show that any non-negative integer N can be written uniquely in the form

$$N = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \cdots + \binom{n_1}{1},$$

where $0 \leqslant n_1 < n_2 < \cdots < n_{k-1} < n_k$.

(Hint: Let n be the value such that $\binom{n}{k} \leqslant N < \binom{n+1}{k}$. Then any possible representation has $n_k = n$. [Why?] Then, use induction and that $N - \binom{n}{k} < \binom{n}{k-1}$ [Why is this?] to show the existence and uniqueness of the representation.)

13. (5 points) Let $n = 2k$ be an even number and $\mathcal{S}$ be a set of n elements. Define a *factor* to be a partition of $\mathcal{S}$ into k sets of size 2. Show that the number of factors is equal to $1 \cdot 3 \cdot 5 \cdots (2k-1)$.

Once you have finished, please write, in the space provided, "I have neither given nor received help on this test through any means other than using books, notes or asking Ryan Hansen." and sign your name.