MAT 378 Exam #1 Solutions 2011-03-02

1. A group of mn people are to be arranged into m teams each with n players.

   (a) (5 points) Determine the number of ways if each team has a different name.

   Solution.
   Two ways. For each, I've used the team numbers as labels.
   Method #1: Line the people up in any order you desire. Then take a permutation of the team numbers $1, 2, \ldots, n$ (or the # of permutations of the multiset $\{n \cdot 1, n \cdot 2, \ldots, n \cdot n\}$).

   $$\frac{(mn)!}{(n!)^m}.$$

   Method #2: Choose each of the teams one by one.

   $$\binom{mn}{n}\binom{(m-1)n}{n}\binom{(m-2)n}{n}\cdots\binom{1 \cdot n}{n}.$$

   ■

   (b) (5 points) Determine the number of ways if the teams don't have names.

   Solution.
   This is the solution to part (a) removing the ordering/labels on the teams (ie. dividing by $m!$).

   $$\frac{(mn)!}{m!(n!)^m}.$$

   ■

2. (4 points) Find the coefficient of $x^{15}y^{12}$ in the expansion of $(4x^3 - 3y^2)^{11}$.

   Solution.
   We pick the $4x^3$ term 5 times to get x^{15} and the $-3y^2$ term 6 times to get y^{12}. This means that $\binom{11}{5}4^5(-3)^6 = \binom{11}{6}4^5(-3)^6$ is the coefficient of $x^{15}y^{12}$ in the expansion. ■

3. (6 points) Determine the number of ways to distribute 10 orange drinks, 1 lemon drink and 1 lime drink to 5 (very) thirsty students (Sarah, Clara, Tucker, Daryl, and Joe) so that each student gets at least one drink and the lemon and lime drinks go to different students.

   Solution.
   Temporarily assign Sarah and Clara the lemon and lime drinks (the counting argument is the same for each of the $5 \cdot 4$ choices of lemon/lime distribution).

   We now have to distribute the $S + C + T + D + J = 10$ orange drinks ensuring that everyone has at least one drink. Since Sarah and Clara already have a drink, we want the number of solutions to $S + C + T' + D' + J' = 7$ in nonnegative integers.

   This means we have $\binom{11}{7} = \binom{11}{4}$ ways to distribute the remaining 7 orange drinks giving $5 \cdot 4 \cdot \binom{11}{7} = 2 \cdot \binom{5}{2}\binom{11}{7}$ ways to distribute all drinks. ■

4. Simplify

   (a) (4 points) $5^{20} - \binom{20}{1}5^{19}\cdot 3^1 + \binom{20}{2}5^{18}\cdot 3^2 - \binom{20}{3}5^{17}\cdot 3^3 + \cdots + \binom{20}{20}3^{20}$.

   Solution.
   This is equivalent to either $(5-3)^{20}$ or $(3-5)^{20}$ which means this is really 2^{20}. ■

   (b) (4 points) $\binom{30}{30} + \binom{30}{29} + \binom{30}{28} + \cdots + \binom{30}{1} + \binom{30}{0}$

   Solution.
   This is 2^{30}. ■

5. Prove $\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$ two ways:

   (a) (5 points) Using Pascal's identity.

   NO CREDIT WILL BE GIVEN FOR SOLUTIONS USING FACTORIALS!

   Solution.

$$\underbrace{\binom{n-2}{k-2} + \binom{n-2}{k-1}}_{\binom{n-1}{k-1}} + \underbrace{\binom{n-2}{k-1} + \binom{n-2}{k}}_{\binom{n-1}{k}} = \binom{n}{k}.$$

■

   (b) (5 points) By giving a combinatorial argument.
   (Hint: Consider a and b as special elements)

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   Solution.
   The LHS counts the number of subsets of size k of a set of size n.

   The RHS considers two special elements a and b. The first binomial represents those subsets of size k that have both of the special elements; the second, those that have only one of the special elements (one for each); the third. those that do not have either of the special elements. ■

6. (a) (6 points) Consider the Boolean lattice when $n = 14$, how many symmetric chains of length 9 go through the set $\{4, 5, 8, 9, 10\}$?

   Solution.
   A symmetric chain of length 9 in this boolean lattice will start at level 3 and go up to level 11 (7 is the middle, it goes up and down 4 levels from there). The set we are given is size 5, so we have to go down two levels and up 6 levels to build a symmetric chain. To go down, we choose the elements to eliminate. To go up, we choose the elements to add, since there are 14 total and we have 5 accounted for that leaves 9 to choose from. This gives a total of

$$\underbrace{5 \cdot 4}_{\text{down}} \cdot \underbrace{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}_{\text{ways up}}.$$

■

   (b) (3 points) How many chains use $\{6, 7, 8, 12, 13\}$ in a symmetric chain decomposition of the Boolean lattice when $n = 14$?

   Solution.
   Since this is a symmetric chain ***decomposition*** (a.k.a partition), every set will be in precisely 1 symmetric chain. ■

7. (6 points) Find an expression for the number of ways to form a committee of 7 people including a chairman from a group of 14 men and 16 women if all of the following must hold:

   1. A man must be the chair of the committee.
   2. There must be a total of at least two men on the committee.
   3. There must be a total of at least three women on the committee.

   (Note: The same 7 people with a different chairman is considered a different committee.)

   Solution.
   There are either 2, 3 or 4 men on the committee and the rest will be women, so

$$2 \cdot \binom{14}{2}\binom{16}{5} + 3 \cdot \binom{14}{3}\binom{16}{4} + 4 \cdot \binom{14}{4}\binom{16}{3}.$$

■

8. (a) (3 points) What does $K_{332} \longrightarrow K_3, K_3, K_3, K_3, K_3$ mean?

Solution.
If we color the edges of a complete graph on 332 vertices red, blue, green, yellow and pink, there must be at least one K_3 (triangle) that is entirely made up of red, blue, green, yellow or pink. ■

(b) (4 points) Using "arrow notation" (partially exemplified when I asked part (a)), explain fully what the notation $R(3, 9) = 36$ means.

Solution.

$$K_{36} \longrightarrow K_3, K_9$$
$$K_{35} \not\longrightarrow K_3, K_9$$

■

(c) (6 points) Use the fact that $51 \leq R(3, 3, 3, 3) \leq 62$ to show that $K_{307} \longrightarrow K_3, K_3, K_3, K_3, K_3$.

Solution.
Choose a special vertex v. Notice, there are 306 edges adjacent to v. The PHP guarantees that if these 306 edges are colored with 5 colors, at least one color will have to be chosen at least 62 times. WLOG, let this color be red. We now consider the complete graph formed by the vertices adjacent to v by red edges. Since $R(3, 3, 3, 3) \leq 62$, we know that if we don't use a red edge (which will create a red K_3), then we will have at least one K_3 of at least one of the other 4 colors. ■