GUIDELINES: You may use books and notes. You can't discuss the problems with anyone other than myself. Make sure to read, write and sign the last page. Answer the questions in the spaces provided. If you run out of room for an answer, continue on the back of the page. There are 8 questions for a total of 100 points.

Name: _

1. (10 points) Show that every natural number can be written as a sum of distinct Fibonacci numbers.

(*Hint:* Use [strong] induction.)

2. (12 points) Let B_n be the number of equivalence relations on n elements. For example, one possible equivalence relation on $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ could be given by

$$[a] = \{2, 3, 7\}, [b] = \{0, 4, 9, 10\}, [c] = \{1\}, [d] = \{5, 6, 8\}$$

Give a combinatorial argument that justifies that B_n satisfies the recurrence

$$B_n = \sum_{k=1}^n \binom{n-1}{k-1} B_{n-k}, \quad B_0 = 1.$$

(*Hint:* Consider a special element x.)

No credit will be given for using recurrences of B_n or factorials.

3. (10 points) We have shown

$$1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n} = n \cdot 2^{n-1}$$

several different ways, but have not done a combinatorial proof yet! Give a combinatorial proof for this equality.

(*Hint:* We might think about committees.)

NO CREDIT WILL BE GIVEN FOR SOLUTIONS USING FACTORIALS!

4. (a) (12 points) Let $p_{n,k}$ denote the number of partitions of n having k parts (not necessarily distinct). Prove that $p_{n,k} = p_{n-1,k-1} + p_{n-k,k}$ and specify the initial values of the recurrence so that this recurrence can be used to determine all other values of $p_{n,k}$.

(*Hint:*) Partition the set of Ferrers diagrams into two sets with the desired sizes.

(b) (3 points) Write the compact form for the generating function for p_{n,k}.
(*Hint:* Write this as a product using ∏ notation. You don't need the solution to problem 4(a) to complete this.)

- 5. Suppose a clock falls off of a wall in a combinatorics class and all of the numbers fall off of the clock. Since the class is filled with clever combinatorics students that want to mess with people, the numbers are replaced in a random order around the circular face (possibly the class is feeling kind and puts them back the way they started, likely not).
 - (a) (6 points) Prove that some set of three consecutive numbers sum to at least 20.

(b) (6 points) Prove that some set of five numbers has sum at least 33.

(c) (12 points) Show that it is <u>not possible</u> for <u>all</u> sets of three consecutive numbers to sum to 19 or 20.

(*Hint:* Think about two consecutive triples first. Then use this to look at numbers opposite other [across the diameter] on the clock face.)

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6. (10 points) A private club has 90 rooms and 100 members. Keys must be given to members such that each set of 90 members can be assigned to 90 distinct rooms whose doors they can open (ie. one person in each room). Each key opens precisely one door. For security purposes, the management would like to minimize the number of keys. Show that the minimum number of keys is 990.

(*Hint:* Consider the scheme where 90 of the members have one key and the remaining 10 have keys to all 90 rooms. Show that this works and fewer keys cannot work.)

- 7. A bridge hand consists of 13 cards from a standard deck of 52 cards.
 - (a) (6 points) What is the probability that a bridge hand has at least one card in each suit?

(b) (3 points) What is the probability that a bridge hand has no cards (a void) in at least one suit?

8. (10 points) There are *n* married couples at a party. They decide that it would be fun to get on a merry-go-round so that no person sits next to their spouse. How many seating arrangements are there that satisfy their requirements? (Note that the seats on a merry-go-round are exactly the same.)

Once you have finished, please write, in the space provided, "I have neither given nor received help on this test through any means other than using books, notes or asking Ryan Hansen." and sign your name.