Name: ____________________________

Math 378 Spring 2011
Bonus Questions 5

1. Let $\mathcal{S} = \{1, 2, \ldots, n\}$. One way to think about a permutation of $\mathcal{S}$ is the following chart:

1	2	3	4	5	6	7	8	9	10
3	1	2	5	4	9	10	8	7	6

We can think of a permutation as a bijective function $f : \mathcal{S} \to \mathcal{S}$. A cycle of a permutation is represented as a grouping of the elements that have changed position with each other (these could be thought of as "mini-permutations" or "sub-permutations"). Using the permutation above as an example, the cycles would be (1 3 2), (4 5), (6 9 7 10) and (8). The length of these cycles is the number of elements in each cycle (ie. cycle lengths 3, 2, 4 and 1).

(a) Let $I(n)$ be the number of permutations having the property that all of its cycles have length 1 or 2. Find a recurrence relation for $I(n)$.

Some values for $I(n)$ are: $I(1) = 1$, $I(2) = 2$. $I(3) = 4$ is given by the permutations (written in cycle form): (1)(2)(3), (1 2)(3), (1 3)(2) and (2 3)(1).

(***Hint***: Split $I(n)$ in two, permutations that fix n and permutations that don't fix n.)

(b) Prove that $I(n)$ is even for all $n > 1^{\dagger}$.

(c) Prove that $I(n) > \sqrt{n!}$ for all $n > 1$.

(***Hint***: For this, you will need the fact that $(1 + \sqrt{n-1}) > \sqrt{n}$. Tell me why this holds.)

(d) What does (c) say about $\dfrac{I(n)}{n!}$?

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†This is a special case of a general fact from group theory where our group is $G = \mathrm{Sym}(n)$. *If G is a finite group of even order, the number of solutions of $x^2 = 1$ is even.*. This is because the elements f for which $f^2 = 1$ come in pairs $\{f, f^{-1}\}$. I've used f above, and this is because we can think of $\mathrm{Sym}(n)$ as a group of bijective functions as well as permutations of $\{1, 2, \ldots, n\}$.