Math 378 Spring 2011
Assignment 5
Solutions

Brualdi 7.1.

Solution.

Conjecture (a)**.** $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$

Proof. ***[By Induction]*** Base case: Let $n = 1$, $f_1 = 1 = f_2$. Now, assume that $f_1 + f_3 + \cdots + f_{2N-1} = f_{2N}$ for all $N < n$. Note that

$$\begin{aligned}
f_{2n} &= f_{2n-1} + f_{2n-2} \\
f_{2n} &= f_{2n-1} + f_{2(n-1)} \\
f_{2n} &= f_{2n-1} + \underbrace{f_{2(n-1)-1} + \cdots + f_3 + f_1}_{\text{By inductive hypothesis}} \\
f_{2n} &= f_{2n-1} + f_{2n-3} + \cdots + f_3 + f_1.
\end{aligned}$$

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Conjecture (b)**.** $f_0 + f_2 + \cdots + f_{2n} = f_{2n+1} - 1$

Proof. ***[By Induction]*** Base case: Let $n = 0$, then $f_0 = 0 = f_1 - 1$. Now, assume that $f_0 + f_2 + \cdots + f_{2N} = f_{2N+1} - 1$ for all $N < n$. Note that

$$\begin{aligned}
f_{2n+1} - 1 &= f_{2n} + f_{2n-1} - 1 \\
f_{2n+1} &= f_{2n} + f_{2(n-1)+1} - 1 \\
f_{2n+1} &= f_{2n} + \underbrace{f_{2(n-1)} + \cdots + f_2 + f_0}_{\text{By inductive hypothesis}} \\
f_{2n+1} &= f_{2n} + f_{2n-2} + \cdots + f_2 + f_0.
\end{aligned}$$

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Brualdi 7.2.

Solution.

Recall, $f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$. Notice that $s_n = \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n < \frac{1}{2}$ for $n \geqslant 0$. This is because $s_0 < \frac{1}{2}$ and $\frac{1-\sqrt{5}}{2} < 1$.

This means that $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$ is less than $\frac{1}{2}$ away from f_n, so f_n is the closest integer.

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Brualdi 7.3.

Solution.

*(a). **[By Induction]*** Note that $f_1 = 1, f_2 = 1$ and $f_3 = 2$ satisfying the base cases for induction. Now, assume that f_n is even $\iff$ N is divisible by 3 for all $N < n$.

Method 1:

Suppose $3 \mid n$. Then $3 \nmid n-1$ and $3 \nmid n-2$. This means that f_{n-1} and f_{n-2} are odd, so $f_n = f_{n-1} + f_{n-2}$ is even.

Suppose $3 \nmid n$. Then either $3 \mid n-1$ or $3 \mid n-2$ but not both. This means either f_{n-1} or f_{n-2} is even and the other is odd which implies that $f_n = f_{n-1} + f_{n-2}$ is odd (since it is the sum of an odd and even number).

Method 2:

Note that

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} \\ f_n &= f_{n-2} + f_{n-3} + f_{n-2} \\ f_n &= 2f_{n-2} + f_{n-3} \qquad (\star) \end{aligned}$$

Now we have $2 \mid f_n \iff 2 \mid (2f_{n-2} + f_{n-3}) \iff 2 \mid f_{n-3}$. ■

*(b). **[By Induction]*** Note that $f_1 = 1, f_2 = 1, f_3 = 2$ and $f_4 = 3$ satisfying the base cases for induction. Now, assume that f_n is divisible by 3 $\iff$ $4 \mid N$ for all $N < n$.

Method 1:

We will consider the Fibonacci numbers modulo 4 for convenience (not necessary but it makes the argument easier/shorter). Note that $f_1 \equiv 1 \pmod 3), f_2 \equiv 1 \pmod 3), f_3 \equiv 2 \pmod 3)$ and $f_4 \equiv 0 \pmod 3)$ and the inductive hypothesis is $f_n \equiv 0 \pmod 3) \iff 4 \mid N$ for all $N < n$. We now proceed similarly to Method 1 above, but the argument is more involved/complicated.

Method 2:

Continuing from $(\star)$ above we have

$$\begin{aligned} f_n &= 2f_{n-2} + f_{n-3} \qquad (\star) \\ f_n &= 2(f_{n-3} + f_{n-4}) + f_{n-3} \\ f_n &= 3f_{n-3} + f_{n-4} \end{aligned}$$

Note that $3 \mid f_n \iff 3 \mid (3f_{n-2} + f_{n-4}) \iff 3 \mid f_{n-4}$. ■

Bruualdi 7.8.

Solution.

Let h_n be the number of ways of coloring a $1 \times n$ chessboard where each square is red or blue and no two adjacent squares are colored red. The first square of the coloring must be either red or blue.

If it is blue, then the number of ways of coloring the other $n-1$ squares is h_{n-1}.

If it is red, then the next square must be blue since two reds cannot be adjacent. There are then h_{n-2} ways to color the rest.

This means the recurrence relation is $h_n = h_{n-1} + h_{n-2}$ for $n \geqslant 2$ with $h_0 = 1$ and $h_1 = 2$. ■

Brualdi 7.9.

Solution.

Let h_n be the number of ways of coloring a $1 \times n$ chessboard where each square is red, white or blue and no two adjacent squares are colored red. The first square of the coloring must be either red, white or blue.

If it is blue, then the number of ways of coloring the other $n-1$ squares is h_{n-1}.

If it is white, then it is also h_{n-1}.

If it is red, then the next square must be either blue or white (2 options) and the rest can be filled in h_{n-2} ways.

This means the recurrence relation is $h_n = 2h_{n-1} + 2h_{n-2}$ for $n \geqslant 2$ with $h_0 = 1$ and $h_1 = 3$. ■

Brualdi 7.10.

Solution.

This is equivalent to: "Use the recurrence $g_n = g_{n-1} + g_{n-2}$ and initial values $g_0 = 0$ and $g_1 = 2$. Find g_{13}. More generally, find f_n."

<u>Method 1:</u>

I'll find a function for g_n and use it to recover g_{13} although you can use the recurrence to do g_{13} pretty reasonably.

To solve this recurrence, I use exactly the method I showed for f_n in class. First, we write the recurrence as $g_n - g_{n-1} - g_{n-2} = 0$ for $n \geqslant 2$. Now, the characteristic equation is $x^2 - x - 1 = 0$ which (according to the quadratic formula) has solutions

$$q_1 = \frac{1+\sqrt{5}}{2}, \qquad q_2 = \frac{1-\sqrt{5}}{2}$$

So we know that

$$g_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Utilizing our initial values we find

$$0 = c_1 + c_2, \qquad 2 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right)$$

Note that $c_2 = -c_1$, so replacing this into the second equation we find

$$\begin{aligned}
2 &= c_1\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) \\
2 &= c_1\sqrt{5} \\
\frac{2}{\sqrt{5}} &= c_1 \implies c_2 = -\frac{2}{\sqrt{5}} \\
\text{So, } g_n &= \frac{2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.
\end{aligned}$$

This means that $g_{13} = 466$.

Method 2:

Note that $g_0 = 0 = 2f_0, g_1 = 2 = 2f_1$ and the recurrence is the same as f_n. This means that $g_n = 2f_n$ which is easily shown by induction (we've done the base cases already). Assume $g_N = 2f_N$ for all $N < n$. Observe,

$$\begin{aligned}
g_n = g_{n-1} + g_{n-2} &= 2f_{n-1} + 2f_{n-2} \\
&= 2(f_{n-1} + f_{n-2}) \\
&= 2f_n.
\end{aligned}$$

Now we already know $f_{13} = 233$, so $g_{13} = 466$. Also, we know that

$$\begin{aligned}
f_n &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \\
\text{So, } g_n = 2f_n &= \frac{2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.
\end{aligned}$$

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Brualdi 7.13.

Solution.

(a)

$$\begin{aligned}
g(x) &= 1 + cx + c^2x^2 + \cdots + c^nx^n + \cdots \\
&= \sum_{n=0}^{\infty} (cx)^n \\
&= \frac{1}{1-cx}
\end{aligned}$$

(b)
$$\begin{aligned} g(x) &= 1 - x + x^2 - x^3 + x^4 - \cdots + (-1)^n x^n + \cdots \\ &= \sum_{n=0}^{\infty} (-x)^n \\ &= \frac{1}{1+x} \end{aligned}$$

(c)
$$\begin{aligned} g(x) &= \binom{\alpha}{0} - \binom{\alpha}{1} x + \binom{\alpha}{2} x^2 - \cdots + (-1)^n \binom{\alpha}{n} x^n + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} x^n \\ &= (1-x)^\alpha \end{aligned}$$

(d)
$$\begin{aligned} g(x) &= 1 + \frac{1}{1!} x + \frac{1}{2!} x^2 + \cdots + \frac{1}{n!} x^n + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n \\ &= e^x \end{aligned}$$

(e)
$$\begin{aligned} g(x) &= 1 - \frac{1}{1!} x + \frac{1}{2!} x^2 - \cdots + (-1)^n \frac{1}{n!} x^n + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n \\ &= e^{-x} \end{aligned}$$

■

Brualdi 7.14.

Solution.

(a)
$$\begin{aligned} g(x) &= \left(x + x^3 + x^5 + \cdots\right)^4 \\ &= \left(x(1 + x^2 + x^4 + \cdots)\right)^4 \\ &= \left(\frac{x}{1-x^2}\right)^4 \end{aligned}$$

(b)
$$\begin{aligned} g(x) &= \left(1 + x^3 + x^6 + \cdots\right)^4 \\ &= \left(\frac{1}{1-x^3}\right)^4 \end{aligned}$$

(c)
$$\begin{aligned} g(x) &= (1)(1+x)\left(1+x+x^2+\cdots\right)^2 \\ &= \frac{1+x}{(1-x)^2} \end{aligned}$$

(d)
$$\begin{aligned} g(x) &= (x+x^3+x^{11})(x^2+x^4+x^5)(1+x+x^2+\cdots)^2 \\ &= \frac{(x+x^3+x^{11})(x^2+x^4+x^5)}{(1-x)^2} \end{aligned}$$

(e)
$$\begin{aligned} g(x) &= (x^{10}+x^{11}+x^{12}+\cdots)^4 \\ &= \frac{x^{40}}{(1-x)^4} \end{aligned}$$

■

Brualdi 7.16.

Solution.
Multiple solutions accepted. ■

Brualdi 7.17.

Solution.
The generating function is given by

$$\begin{aligned} g(x) &= (1+x^2+x^4+\cdots)(1+x+x^2)(1+x^3+x^6+x^9+\cdots)(1+x) \\ &= \frac{1}{1-x^2}\cdot\frac{1-x^3}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1-x^2}{1-x} \\ &= \frac{1}{(1-x)^2} \end{aligned}$$

Recall, Newton's Binomial Theorem (as well as an early example of generating functions that I did in class) gives that

$$\begin{aligned} \frac{1}{(1-z)^n} = (1-z)^{-n} &= \sum_{k=0}^{\infty}\binom{n+k-1}{k}z^k \\ \implies g(x) &= \sum_{k=0}^{\infty}\binom{2+k-1}{k}x^k \end{aligned}$$

We now want the coefficient of x^n so we set $k=n$ and get $\binom{n+1}{n} = n+1$. Thus $h_n = n+1$. ■

Brualdi 7.22.

Solution.

$$\begin{aligned} g^{(e)}(x) &= 0! + 1!\frac{x^1}{1!} + 2!\frac{x^2}{2!} + 3!\frac{x^3}{3!} + \cdots + n!\frac{x^n}{n!} + \cdots \\ &= 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \\ &= \frac{1}{1-x} \end{aligned}$$

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Brualdi 7.24.

Solution.

(a)

$$g^{(e)}(x) = \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)^k$$

(b)

$$\begin{aligned} g^{(e)}(x) &= \left(\frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots\right)^k \\ &= \left(e^x - \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right]\right)^k \end{aligned}$$

(c)

$$\begin{aligned} g^{(e)}(x) &= \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\cdots\left(\frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!}\cdots\right) \\ &= (e^x - 1)(e^x - 1 - x)\cdots\left(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^{k-1}}{(k-1)!}\right) \\ &= \prod_{i=0}^{k-1}\left(e^x - \sum_{j=0}^{i}\frac{x^j}{j!}\right) \end{aligned}$$

(d)

$$\begin{aligned} g^{(e)}(x) &= (1+x)\left(1 + x + \frac{x^2}{2!}\right)\cdots\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}\right) \\ &= \prod_{i=1}^{k}\sum_{j=0}^{i}\frac{x^j}{j!} \end{aligned}$$

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Brualdi 7.26.

Solution.

The exponential generating function for each piece is

$$\begin{aligned} \text{Red: } & 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots & \text{Red: } & \frac{e^x+e^{-x}}{2} \\ \text{Blue: } & 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots & \text{Blue: } & e^x \\ \text{Green: } & 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots & \text{Green: } & \frac{e^x+e^{-x}}{2} \\ \text{Orange: } & 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots & \text{Orange: } & e^x \end{aligned}$$

Thus the exponential generating function

$$g^{(e)}(x) = e^{2x}\left(\frac{e^x+e^{-x}}{2}\right)^2$$
$$g^{(e)}(x) = \frac{e^{2x}+2e^x+1}{4}$$

■

Brualdi 7.32.

Solution.

Iterating the recurrence yields

$$\begin{aligned} h_n &= (n+2)h_{n-1} \\ &= (n+2)(n+1)h_{n-2} \\ &= (n+2)(n+1)\cdots 4\cdot 3h_0 \\ &= (n+2)(n+1)\cdots 3\cdot 2 \\ &= (n+2)! \end{aligned}$$

■

Brualdi 7.33.

Solution.

Rewrite the recurrence as $h_n - h_{n-1} - 9h_{n-2} + 9h_{n-3},\ n \geqslant 3$ and $h_0 = 0, h_1 = 1$ and $h_2 = 2$. This means the characteristic equation is

$$x^3 - x^2 - 9x + 9 = 0$$
$$x^2(x-1) - 9(x-1) = 0$$
$$(x+3)(x-3)(x-1) = 0$$

So the general solution is

$$h_n = c_1 + c_2 \cdot 3^n + c_3 \cdot (-3)^n$$

Utilizing our initial values we have the following system of equations

$$\begin{aligned} 0 &= c_1 + c_2 + c_3 && (1) \\ 1 &= c_1 + 3c_2 - 3c_3 && (2) \\ 2 &= c_1 + 9c_2 + 9c_3 && (3) \end{aligned}$$

Adding -9 times (1) to (3) we get $c_1 = -\frac{1}{4}$. Adding 3 times (2) to (3) and using our value for c_1 gives us that $c_2 = \frac{2}{3}$. Finally substituting into (1) we get that $c_3 = \frac{1}{4}$.

This means a general formula is

$$h_n = -\frac{1}{4} + 2 \cdot 3^{n-1} + \frac{(-3)^n}{4}.$$

■

Bruualdi 7.37.

Solution.

Let h_n denote the number of ternary strings of length n that do not contain two consecutive 0's nor two consecutive 1's. The initial values are $h_0 = 1$ and $h_1 = 3$.

Method 1:

Suppose the string starts with a 2, there are h_{n-1} ways to finish the rest.

Suppose the string starts with 0 or 1 (2 choices), and the second character is a 2. There are h_{n-2} ways to finish this making $2h_{n-2}$ strings for this case.

Suppose the string starts with 0 or 1, the second character is 1 or 0 (resp.) (two total choices), and the third is a 2. There are h_{n-3} ways to finish this making $2h_{n-3}$ strings like this.

Continue until you have the case where the string is all 0s and 1s. There are two such strings (two choices and h_0 ways to finish them).

The recurrence we have found is $h_n = h_{n-1} + 2h_{n-2} + 2h_{n-3} + \cdots + 2h_0$ which seems to be a disaster, however, note that

$$\begin{aligned} h_n &= h_{n-1} + 2h_{n-2} + 2h_{n-3} + \cdots + 2h_0 \\ -h_{n-1} &= \qquad\qquad -h_{n-2} - 2h_{n-3} - \cdots - 2h_0 \\ h_n - h_{n-1} &= h_{n-1} + h_{n-2} \\ h_n &= 2h_{n-1} + h_{n-2} \end{aligned}$$

We now see the characteristic equation is $x^2 - 2x - 1 = 0 \implies (x-1)^2 = 0$ and the quadratic formula gives $1 - \sqrt{2}$ and $1 + \sqrt{2}$ as the roots. Thus we have the general solution is

$$h_n = c_1 \left(1 - \sqrt{2}\right)^n + c_2 \left(1 + \sqrt{2}\right)^n$$

Utilizing our initial conditions we find the following system of equations

$$1 = c_1 + c_2 \tag{4}$$
$$3 = c_1\left(1-\sqrt{2}\right) + c_2\left(1+\sqrt{2}\right) \tag{5}$$

Solving this system we find that $c_1 = \frac{1-\sqrt{2}}{2}$ and $c_2 = \frac{1+\sqrt{2}}{2}$. Therefore our formula is

$$\begin{aligned} h_n &= \left(\frac{1-\sqrt{2}}{2}\right)\left(1-\sqrt{2}\right)^n + \left(\frac{1+\sqrt{2}}{2}\right)\left(1+\sqrt{2}\right)^n \\ &= \frac{1}{2}\left(1-\sqrt{2}\right)^{n+1} + \frac{1}{2}\left(1+\sqrt{2}\right)^{n+1}. \end{aligned}$$

Method 2:

Let a_n denote the ternary strings of length n ending in a 0 (similarly a_n will also denotes the ternary strings ending in a 1 since there are the same number of them). Let b_n denote the ternary strings of length n ending in a 2. The initial values are $h_0 = 0$ and $a_1 = 1, a_2 = 2, b_1 = 1$ and $b_2 = 3$.

If a string is length $n-1$ and ends in a 0, it will generate two strings of length n, one ending in 2, the other in 1.

If a string is length $n-1$ and ends in a 1, it will generate two strings of length n, one ending in 2, the other in 0.

If a string is length $n-1$ and ends in a 2, it will generate three strings of length n, one each ending in 0, 1 and 2.

The initial values are $h_0 = 0$ and $a_1 = 1, a_2 = 2, b_1 = 1$ and $b_2 = 3$.

n	End in 0 (a_n)	End in 1 (a_n)	End in 2 (b_n)	h_n
1	1	1	1	3
Contrib. of 0's	0	1	1	
Contrib. of 1's	1	0	1	
Contrib. of 2's	1	1	1	
2	2	2	3	7
Contrib. of 0's	0	2	2	
Contrib. of 1's	2	0	2	
Contrib. of 2's	3	3	3	
3	5	5	7	17
Contrib. of 0's	0	5	5	
Contrib. of 1's	5	0	5	
Contrib. of 2's	7	7	7	
4	12	12	17	41
Contrib. of 0's	0	12	12	
Contrib. of 1's	12	0	12	
Contrib. of 2's	17	17	17	
5	29	29	41	99

There are a few things made clear by these arguments. First, we have a recurrence for a_n and b_n as follows:

$$\begin{aligned} a_n &= a_{n-1} + b_{n-1} \\ b_n &= 2a_{n-1} + b_{n-1} \end{aligned}$$

Notice that $h_n = 2a_n + b_n$ since this is the disjoint sum of the strings ending in 0, 1 and 2 respectively. Surprisingly (perhaps) $h_{n-1} = 2a_{n-1} + b_{n-1} = b_n$. Now, let's find a recurrence for h_n.

$$\begin{aligned} h_n &= 2a_n + b_n \\ &= 2(a_{n-1} + b_{n-1}) + b_n \\ &= (2a_{n-1} + b_{n-1}) + b_{n-1} + b_n \\ &= 2b_n + b_{n-1} \\ &= 2h_{n-1} + h_{n-2}. \end{aligned}$$

Solving the recurrence is the same as in ***Method 1***. ■

Brualdi 7.39.

Solution.

Let h_n be the number of ways to perfectly cover a $1 \times n$ board with monominoes and dominoes in such a way that no two dominoes are consecutive. The initial conditions are $h_0 = 1$ and $h_1 = 1$. Any covering of a $1 \times n$ board must start with either a monomino or a domino.

Suppose it starts with a monomino, then there are h_{n-1} ways to finish the remaining $1 \times n - 1$ board.

Now, suppose it starts with a domino, then the next square must be covered by a monomino since there cannot be two consecutive dominoes. This means there are h_{n-3} ways to finish covering the $1 \times n - 3$ board that is left.

This means the recurrence is $h_n = h_{n-1} + h_{n-3}$ with $h_0 = 1, h_1 = 1$ and $h_2 = 2$.

In case you were wondering why we don't want to solve this, the solution is on the next page. ■

This is the solution to the recurrence in the last problem (**Brualdi 7.39**).

$$\begin{aligned}
h_n = &\left(\frac{1}{3} + \frac{1}{93}\sqrt[3]{\frac{50933}{22} - \frac{4371\sqrt{93}}{2}} + \frac{\sqrt[3]{\frac{1}{2}\left(1643 + 141\sqrt{93}\right)}}{3 \cdot 31^{2/3}}\right) \\
&\quad \cdot \left(\frac{1}{3} + \frac{1}{3}\sqrt[3]{\frac{29}{2} - \frac{3\sqrt{93}}{2}} + \frac{1}{3}\sqrt[3]{\frac{1}{2}\left(29 + 3\sqrt{93}\right)}\right)^n \\
&+ \left(\frac{1}{3} - \frac{1}{186}(1 + i\sqrt{3})\sqrt[3]{\frac{50933}{2} - \frac{4371\sqrt{93}}{2}} - \frac{\left(1 - i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(1643 + 141\sqrt{93}\right)}}{6 \cdot 31^{2/3}}\right) \\
&\quad \cdot \left(\frac{1}{3} - \frac{1}{6}\left(1 + i\sqrt{3}\right)\sqrt[3]{\frac{29}{2} - \frac{3\sqrt{93}}{2}} - \frac{1}{6}\left(1 - i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(29 + 3\sqrt{93}\right)}\right)^n \\
&+ \left(\frac{1}{3} - \frac{1}{186}\left(1 - i\sqrt{3}\right)\sqrt[3]{\frac{50933}{2} - \frac{4371\sqrt{93}}{2}} - \frac{\left(1 + i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(1643 + 141\sqrt{93}\right)}}{6 \cdot 31^{2/3}}\right) \\
&\quad \cdot \left(\frac{1}{3} - \frac{1}{6}\left(1 - i\sqrt{3}\right)\sqrt[3]{\frac{29}{2} - \frac{3\sqrt{93}}{2}} - \frac{1}{6}\left(1 + i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(29 + 3\sqrt{93}\right)}\right)^n
\end{aligned}$$

Brualdi 7.52.

Solution.

(b) Iterating the recurrence yields

$$\begin{aligned}
h_n &= 5h_{n-1} + 5^n \\
&= 5(5h_{n-2} + 5^n) + 5^n \\
&= 5^2 h_{n-2} + 5^{n+1} + 5^n \\
&= 5^2(5h_{n-3} + 5^n) + 5^{n+1} + 5^n \\
&= 5^3 h_{n-3} + 5^{n+2} + 5^{n+1} + 5^n \\
&= 5^n h_0 + 5^{2n-1} + 5^{2n-2} + \cdots + 5^{n+1} + 5^n \\
&= 3 \cdot 5^n + 5^n(1 + 5 + 5^2 + \cdots + 5^{n-1}) \\
&= 3 \cdot 5^n + 5^n \cdot \frac{5^n - 1}{4} \\
&= \frac{1}{4}5^n(5^n + 11)
\end{aligned}$$

∎

Extra Problem #1.

Solution.

To find the generating function for the nonnegative integral solutions to $3e_1 + 7e_2 + 3e_3 + 5e_4 = n$, we reformat this as $f_1 + f_2 + f_3 + f_4 = n$ where f_1 is a multiple of 3, f_2 is a multiple of 7, f_3 is a multiple of 3 and f_4 is a multiple of 5. Thus we have that the generating function is

$$\begin{aligned} g(x) &= (1 + x^3 + x^6 + \cdots)(1 + x^7 + x^{14} + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^5 + x^{10} + \cdots) \\ &= \left(\frac{1}{1 - x^3}\right)^2 \frac{1}{1 - x^7} \frac{1}{1 - x^5} \end{aligned}$$

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