Math 378 Spring 2011
Assignment 4
Solutions

Brualdi 6.2.

Solution.
The properties are

$$\begin{aligned} P_1 &: \text{is divisible by 4.} \\ P_2 &: \text{is divisible by 6.} \\ P_3 &: \text{is divisible by 7.} \\ P_4 &: \text{is divisible by 10.} \end{aligned}$$

Preparing to use inclusion-exclusion, we observe that

$$\begin{aligned}
& & |A_1 \cap A_2| &= \left\lfloor \frac{10000}{12} \right\rfloor = 833 & & \\
|A_1| &= \left\lfloor \frac{10000}{4} \right\rfloor = 2500 & |A_1 \cap A_3| &= \left\lfloor \frac{10000}{28} \right\rfloor = 357 & |A_1 \cap A_2 \cap A_3| &= \left\lfloor \frac{1000}{84} \right\rfloor = 119 \\
|A_2| &= \left\lfloor \frac{10000}{6} \right\rfloor = 1666 & |A_1 \cap A_4| &= \left\lfloor \frac{10000}{20} \right\rfloor = 500 & |A_1 \cap A_2 \cap A_4| &= \left\lfloor \frac{1000}{60} \right\rfloor = 166 \\
|A_3| &= \left\lfloor \frac{10000}{7} \right\rfloor = 1428 & |A_2 \cap A_3| &= \left\lfloor \frac{10000}{42} \right\rfloor = 238 & |A_1 \cap A_3 \cap A_4| &= \left\lfloor \frac{1000}{140} \right\rfloor = 71 \\
|A_4| &= \left\lfloor \frac{10000}{10} \right\rfloor = 1000 & |A_2 \cap A_4| &= \left\lfloor \frac{10000}{30} \right\rfloor = 333 & |A_2 \cap A_3 \cap A_4| &= \left\lfloor \frac{1000}{210} \right\rfloor = 47 \\
& & |A_3 \cap A_4| &= \left\lfloor \frac{10000}{70} \right\rfloor = 142 & & \\
& & |A_1 \cap A_2 \cap A_3 \cap A_4| &= \left\lfloor \frac{10000}{420} \right\rfloor = 23 & &
\end{aligned}$$

Thus, by inclusion-exclusion, the number of values between 1 and 10,000 inclusive

$$\begin{aligned}
\left|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}\right| &= 10000 - (2500 + 1666 + 1428 + 1000) + (833 + 357 + 500 + 238 + 333 + 142) \\
&\quad - (119 + 166 + 71 + 47) + 23 \\
&= 5429.
\end{aligned}$$

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Brualdi 6.3.

Solution.

The properties are

$$P_1 : \text{is a perfect square.}$$
$$P_2 : \text{is a perfect cube.}$$

Preparing for inclusion-exclusion, we observe that

$$|A_1| = \left\lfloor \sqrt{10000} \right\rfloor = 100 \qquad |A_1 \cap A_2| = \left\lfloor \sqrt[6]{10000} \right\rfloor = 4$$
$$|A_2| = \left\lfloor \sqrt[3]{10000} \right\rfloor = 21$$

Thus, by inclusion exclusion, we have

$$\left|\overline{A_1} \cap \overline{A_2}\right| = 10000 - (100 + 21) + 4 = 9883.$$

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Brualdi 6.7.

Solution.

The properties are

$$P_1 : x_1 \geq 9.$$
$$P_2 : x_2 \geq 9.$$
$$P_3 : x_3 \geq 9.$$
$$P_4 : x_4 \geq 9.$$

Preparing for inclusion-exclusion we note that $|A_1| = |A_2| = |A_3| = |A_4|$. Also, $|A_1|$ represents the number of nonnegative integer solutions of $x' + x_2 + x_3 + x_4 = 5$. This number is $\binom{8}{5} = 56$. Additionally, no more than one property can hold for a given solution to $x_1 + x_2 + x_3 + x_4 = 14$ in nonnegative integers.

Thus, by inclusion-exclusion, we have

$$\left|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}\right| = \binom{17}{14} - 4 \cdot \binom{8}{5} = 680 - 4 \cdot 56 = 456.$$

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Brualdi 6.9.

Solution.

To account for the lower bounds we realize that this is equivalent to the number of nonnegative integer solutions to $y_1 + y_2 + y_3 + y_4 = 13$ where $y_1 \leq 5, y_2 \leq 7, y_3 \leq 4, y_4 \leq 4$.

The properties are

$$\begin{aligned} P_1 &: y_1 \geq 6. \\ P_2 &: y_2 \geq 8. \\ P_3 &: y_3 \geq 5. \\ P_4 &: y_4 \geq 5. \end{aligned}$$

Preparing for inclusion exclusion, we observe that

$$\begin{aligned} & & |A_1 \cap A_2| &= 0 & & \\ |A_1| &= \binom{10}{7} & |A_1 \cap A_3| &= \binom{5}{2} & & \\ |A_2| &= \binom{8}{5} & |A_1 \cap A_4| &= \binom{5}{2} & |A_i \cap A_j \cap A_k| &= 0 \\ |A_3| &= \binom{11}{8} & |A_2 \cap A_3| &= \binom{3}{0} & |A_1 \cap A_2 \cap A_3 \cap A_4| &= 0 \\ |A_4| &= \binom{11}{8} & |A_2 \cap A_4| &= \binom{3}{0} & & \\ & & |A_3 \cap A_4| &= \binom{6}{3} & & \end{aligned}$$

Thus, by inclusion-exclusion, we have

$$\binom{16}{13} - \left(\binom{10}{7} + \binom{8}{5} + 2 \cdot \binom{11}{8}\right) + \left(2 \cdot \binom{5}{2} + 2 \cdot \binom{3}{0} + \binom{6}{3}\right) = 96.$$

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Brualdi 6.10.

Solution.

Note that this is the same as finding the number of nonnegative integer solutions of $a_1 + a_2 + \cdots + a_k = r$ where $a_i \leq n_i$. Let P_i be that $a_i \geq n_i + 1$. The existence of an r combination means $r \leq n_1 + n_2 + \cdots + n_k$. If a solution satisfies all k properties,

$$\begin{aligned} a_1 + a_2 + \cdots + a_k &\geq (n_1 + 1) + (n_2 + 1) + \cdots + (n_k + 1) \\ &= n_1 + n_2 + \cdots + n_k + k \\ &> r. \end{aligned}$$

Thus, the number of solutions satisfying all k properties is $0 \implies A_1 \cap A_2 \cap \cdots \cap A_k = \emptyset$. ■

Brualdi 6.12.

Solution.

Choose the 4 elements to move: $\binom{8}{4}$ ways. Then derange those elements: D_4 ways. This makes $\binom{8}{4} D_4$ ways total. ■

Brualdi 6.14.

Solution.

By the above argument, $\binom{n}{k} D_{n-k}$. ■

Brualdi 6.15.

Solution.

(a) D_7.

(b) The only thing not allowed is that no gentleman receives his own hat, so $7! - D_7$.

(c) As in (b), except we also must eliminate when only one receives his own hat, so $7! - D_7 - \binom{7}{1} D_6$. ■

Brualdi 6.16.

Solution.

The left hand side counts the number of permutations of $\{1, 2, \ldots, n\}$ in the obvious way.

The right hand side counts the number of permutations of $\{1, 2, \ldots, n\}$ by looking at how many numbers are in the same position as their value. Summing over all possible numbers of positions gives the total number of permutations of $\{1, 2, \ldots, n\}$. ■

Brualdi 6.21.

Solution. (By Induction)

Note that $D_1 = 0$ and $D_2 = 1$ satisfying our base cases for induction. Now assume that D_n is even $\iff$ n is odd for $k < n$.

<u>Method 1:</u>

Consider $D_{k+1} = k(D_{k-2} + D_{k-1})$.

If $k + 1$ is odd, then k is even so D_{k+1} is even.

If $k + 1$ is even, then k is odd. By our inductive hypothesis, D_{k-2} is odd and D_{k-1} is even. This means the sum $D_{k-2} + D_{k-1}$ is odd. Also, we know that an odd number times an odd number is odd, so D_{k+1} is odd.

<u>Method 2:</u>

Note that $\dfrac{n!}{(n-1)!}$ is even $\iff$ n is even and $\dfrac{n!}{k!}$ for $0 \le k < n-1$ is always even (since n and $n-1$ are factors and one must be even). Since

$$D_n = n! - \frac{n!}{1} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^{n-1}\frac{n!}{(n-1)!} + (-1)^n$$

We know that all the values are even except possibly the last two. If n is odd, then we have precisely two odd numbers in the sum, so D_n is even. If n is even, then everything is even except ± 1, so D_n is odd.

<u>Method 3:</u>

Consider $D_n = nD_{n-1} + (-1)^n$. ■

Brualdi 6.29.

Solution.

It is easiest to think of this as giving out tickets: put the people in a line and give out tickets labeled with the stop number. This is exactly a 10-permutation using the numbers 1 through 6. We are interested when the permutation has at least one of each value. The properties for inclusion-exclusion would be:

$$\begin{aligned}
P_1 &: \text{No one gets out at stop 1.}\\
P_2 &: \text{No one gets out at stop 2.}\\
P_3 &: \text{No one gets out at stop 3.}\\
P_4 &: \text{No one gets out at stop 4.}\\
P_5 &: \text{No one gets out at stop 5.}\\
P_6 &: \text{No one gets out at stop 6.}
\end{aligned}$$

Setting the properties this way shows that the set intersections in inclusion-exclusion will be symmetric (ie. no one getting out at stops 1 and 2 will be the same number as stops 3 and 5). Inclusion-exclusion is as follows:

$$\begin{aligned}
|A_i| &= 5^{10}\\
|A_i \cap A_j| &= 4^{10}\\
|A_i \cap A_j \cap A_k| &= 3^{10}\\
|A_i \cap A_j \cap A_k \cap A_l| &= 2^{10}\\
|A_i \cap A_j \cap A_k \cap A_l \cap A_m| &= 1^{10}\\
|A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6| &= 0
\end{aligned}$$

$$\left|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5} \cap \overline{A_6}\right| = 6^{10} - \binom{6}{1}5^{10} + \binom{6}{2}4^{10} - \binom{6}{3}3^{10} + \binom{6}{4}2^{10} - \binom{6}{5}1^{10}.$$

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Extra Problem #1.

Solution.

(a). Number the girls 1 through 8 and assume they face in a counterclockwise direction. The forbidden arrangements have one or more of 1 looking at 2, 2 looking at 3, ..., or 8 looking at 1. Consider the following properties:

$$\begin{aligned} P_{12} &: \text{The rearrangement has 1 looking at 2.} \\ P_{23} &: \text{The rearrangement has 2 looking at 3.} \\ &\vdots \\ P_{81} &: \text{The rearrangement has 8 looking at 1.} \end{aligned}$$

There is an easy and a hard way to think about the rest:

Hard Way: Since the animal seats are distinct, there are $7! \cdot 8 = 8!$ arrangements total. Note that $|A_{12}|$ is equivalent to the number of arrangements of $\left\{\boxed{12}, 3, 4, 5, 6, 7, 8\right\}$, so $8 \cdot 6!$. This is because we look at a circular permutation of these 7 objects (6!) but the starting point matters, so there are 8 of each. This is similar for all $|A_{ij}|$.

Let's think about the possible two way intersections. Consider $|A_{12} \cap A_{34}|$. This is the number of arrangements of $\left\{\boxed{12}, \boxed{34}, 5, 6, 7, 8\right\}$ so a similar argument gives $8 \cdot 5!$. If we consider $|A_{12} \cap A_{23}|$, we have the number of arrangements of $\left\{\boxed{123}, 4, 5, 6, 7, 8\right\}$ or $8 \cdot 5!$ also.

Now for three way intersections. The types of possibilities are as follows:

Consider $|A_{12} \cap A_{34} \cap A_{56}|$. This is the number of arrangements of $\left\{\boxed{12}, \boxed{34}, \boxed{56}, 7, 8\right\}$. This is $8 \cdot 4!$. Consider $|A_{12} \cap A_{23} \cap 45|$. This is the number of arrangements of $\left\{\boxed{123}, \boxed{45}, 6, 7, 8\right\}$ which is $8 \cdot 4!$. Consider $|A_{12} \cap A_{23} \cap A_{34}|$. This is the number of arrangements of $\left\{\boxed{1234}, 5, 6, 7, 8\right\}$ which is $8 \cdot 4!$.

Continue ad nauseam. OR

Easy Way: The "easier" way to think about this is that for each property satisfied, we reduce the number of objects in the circular permutation by one. Thus we have that if we want the number of permutations satisfying k properties, we find the number of circular permutations of $8-k$ objects, ($(8-k-1)!$ ways) and then account for the 8 different starting positions when finding the arrangements. This doesn't work for $k = 8$, but to satisfy all 8 properties there is only one circular permutation, but starting position matters so there are 8 arrangements.

Either way we have (by inclusion-exclusion),

$$
\begin{aligned}
&= 8! - \binom{8}{1} 8 \cdot 6! + \binom{8}{2} 8 \cdot 5! - \binom{8}{3} 8 \cdot 4! + \binom{8}{4} 8 \cdot 3! - \binom{8}{5} 8 \cdot 2! + \binom{8}{6} 8 \cdot 1! \\
&\qquad - \binom{8}{7} 8 \cdot 0! + \binom{8}{8} 8 \cdot 1 \\
&= 8\Big[7! - \binom{8}{1} 6! + \binom{8}{2} 5! - \binom{8}{3} 4! + \binom{8}{4} 3! - \binom{8}{5} 2! + \binom{8}{6} 1! - \binom{8}{7} 0! + \binom{8}{8} 1\Big] \qquad (\star)
\end{aligned}
$$

Note the similarity between these and Q_n (in particular Q_7 and Q_8). ■

(b). If all the seats are identical, we divide $(\star)$ by 8 so we get:

$$
\begin{aligned}
\text{Number of rearrangements } &= 7! - \binom{8}{1} 6! + \binom{8}{2} 5! - \binom{8}{3} 4! \\
&\quad + \binom{8}{4} 3! - \binom{8}{5} 2! + \binom{8}{6} 1! - \binom{8}{7} 0! + \binom{8}{8} 1.
\end{aligned}
$$

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(c). If the girls are facing in a line, this is exactly Q_8 as discussed in class. ■

(d). Q_n. ■

Extra Problem #2.

Solution.

(a). This is a 6×6 board. We can regard the non-attacking rooks as permutations of $\{1, \ldots, 6\}$ where the ith position indicates the row of the rook in column i for $i = 1, \ldots, 6$. The forbidden spots relate to the following properties:

P_{11} : There is a 1 in position 1 of the permutation. (ie. a rook is placed at (1,1)).

P_{12} : There is a 1 in position 2 of the permutation. (ie. a rook is placed at (1,2)).

P_{23} : There is a 2 in position 3 of the permutation.

P_{24} : There is a 2 in position 4 of the permutation.

P_{35} : There is a 3 in position 5 of the permutation.

P_{36} : There is a 3 in position 6 of the permutation.

Further, note that for i, k, m distinct, $j = 1, 2$, $l = 3, 4$, $n = 5, 6$:

$$\begin{aligned} |A_{ij}| &= 5! \\ |A_{ij} \cap A_{kl}| &= 4! \\ |A_{ij} \cap A_{kl} \cap A_{mn}| &= 3! \end{aligned}$$

By inclusion-exclusion, we have

$$\begin{aligned} \text{Number of ways to place the rooks } &= 6! - 6 \cdot 5! + \overbrace{12}^{\binom{3}{2} \cdot 2^2} \cdot 4! - \overbrace{8}^{2^3} \cdot 3! \\ &= 240. \end{aligned}$$

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(b). This is an 8×8 board and we want permutations where 1 is not in position 1, 2 is not be in position 2,. . ., 8 is not be in position 8. This is precisely D_8. ■

(c). The probability of (b) happening is $\dfrac{D_8}{8!} = \dfrac{14833}{40320} = .367881\overline{944}$.

Interestingly, $\lim\limits_{x \to \infty} \dfrac{D_n}{n!} = \dfrac{1}{e} = .\mathbf{36787}944117\ldots$ ■

Extra Problem #3.

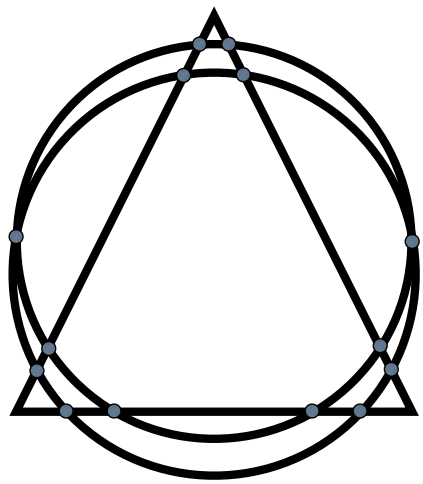
Solution.

Inclusion-exclusion states that we would take the pairwise intersections and subtract the three way intersections. This means that would like to maximize the pairwise intersections while minimizing the three way intersections. Let the figures be C_1, C_2 and T. The maximum

number of intersections for these types of figures (since $C_1 \cap C_2$ is not allowed to be ∞ since the circles are distinguishable) is given below.

$$\max(|C_1 \cap C_2|) = 2$$
$$\max(|C_i \cap T|) = 6, \text{ for } i = 1, 2.$$

If we can realize these maxima while keeping $|C_1 \cap C_2 \cap T| = 0$, then we have clearly found the maximum number of points belonging to at least two of these figures which would be $6 + 6 + 2 = 14$. This can be done as in the picture below.



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