Math 378 Spring 2011
Assignment 3
Solutions

Bruualdi 5.7.

Solution.

Take $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ and set $x = 2, y = 1$. Also, $\sum_{k=0}^{n} r^k = (r+1)^n$. ■

Brualdi 5.8.

Solution.

Setting $y = 3$ and $x = -1$ in the binomial theorem to get $2^n = (3-1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}$. ■

Brualdi 5.9.

Solution.

Setting $x = 1$ and $y = -10$ in the binomial theorem gives $(-9)^n = \sum_{k=0}^{n} \binom{n}{k} (-10)^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 10^k$. ■

Brualdi 5.11.

Solution.

Both sides count the number of k-subsets containing at least one of x, y, and z (ie. subsets of size k that have non-empty intersection with $\{x, y, z\}$) of an n-set containing x, y, and z.

The LHS counts all the k-subsets and removes the number without x, y, z.

For the RHS, $\binom{n-1}{k-1}$ counts the number of k-subsets containing x; $\binom{n-2}{k-1}$ is the number of k-subsets not containing x, but containing y; $\binom{n-2}{k-1}$ is the number of k-subsets containing neither x nor y, but containing z. ■

Bruualdi 5.12.

Solution.

If n is odd, then the terms $(-1)^k\binom{n}{k}^2$ and $(-1)^{n-k}\binom{n}{n-k}^2$ cancel, so the sum is 0.

If n is even, let $n = 2m$. Following the hint, if we expand the LHS by the binomial theorem, we have:

$$\begin{aligned}(1-x^2)^n &= \sum_{k=0}^{n}\binom{n}{k}(-x^2)^k \\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k x^{2k}\end{aligned}$$

and so we can find the coefficient of x^n by setting $2k = n = 2m \implies k = m$ which gives a coefficient of $(-1)^m\binom{2m}{m}$.

If we expand both factors on the RHS by the binomial theorem, and then take their product, we have:

$$(1+x)^n(1-x)^n = \left[\sum_{k=0}^{n}\binom{n}{k}x^n\right]\left[\sum_{j=0}^{n}\binom{n}{k}(-1)^j x^j\right]$$

This gives a coefficient of x^n whenever $k+j=n$, so the coefficient of x^n is

$$\underbrace{\binom{n}{n}(-1)^0\binom{n}{0}}_{k=n,\ j=0} + \underbrace{\binom{n}{n-1}(-1)^1\binom{n}{1}}_{k=n-1,\ j=1} + \cdots + \underbrace{\binom{n}{1}(-1)^{n-1}\binom{n}{n-1}}_{k=1,\ j=n-1} + \underbrace{\binom{n}{0}(-1)^n\binom{n}{n}}_{k=0,\ j=n}$$

$$\begin{aligned}&= (-1)^0\binom{n}{0}^2 + (-1)^1\binom{n}{1}^2 + \cdots + (-1)^n\binom{n}{n}^2 \\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}^2.\end{aligned}$$

■

Bruualdi 5.13.

Solution.

First, the boring derivation.

$$\begin{aligned}\binom{n}{k}+3\binom{n}{k-1}+3\binom{n}{k-2}+\binom{n}{k-3} &= \left[\binom{n}{k}+\binom{n}{k-1}\right]+2\left[\binom{n}{k-1}+\binom{n}{k-2}\right] \\ &\quad+\left[\binom{n}{k-2}+\binom{n}{k-2}\right] \\ &= \binom{n+1}{k}+2\binom{n+1}{k-1}+\binom{n+1}{k-2} \\ &= \left[\binom{n+1}{k}+\binom{n+1}{k-1}\right]+\left[\binom{n+1}{k-1}+\binom{n+1}{k-2}\right] \\ &= \binom{n+2}{k}+\binom{n+2}{k-1} \\ &= \binom{n+3}{k}.\end{aligned}$$

Now for a more exciting combinatorial argument showing the second of the following two equivalences:

$$\binom{n}{k}+3\binom{n}{k-1}+3\binom{n}{k-2}+\binom{n}{k-3}=\binom{n+3}{k}$$

$$\binom{3}{0}\binom{n}{k}+\binom{3}{1}\binom{n}{k-1}+\binom{3}{2}\binom{n}{k-2}+\binom{3}{3}\binom{n}{k-3}=\binom{n+3}{k} \tag{1}$$

The RHS represents the number of k-subsets of a set of size $n+3$.

The LHS counts the number of ways of choosing a k-subset of a set of size $n+3$ with consideration of 3 special elements x, y, and z. Rewriting it in the form of (1) makes the argument easy to see. ■

Brualdi 5.15.

Solution.

<u>Method 1</u>: Using the committee with chairperson idea, $\binom{n}{k}k=n\binom{n-1}{k-1}$, we have

$$\begin{aligned}-\sum_{k=1}^{n}(-1)^k\binom{n}{k}k &= -\sum_{k=1}^{n}(-1)^k n\binom{n-1}{k-1} \\ &= -n\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j \\ &= 0.\end{aligned}$$

We have the last equality because the sum is alternating signs of a row of Pascal's Triangle.

***Method 2*:** We use the binomial theorem as follows:

$$\begin{aligned} (1+x)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k \\ \frac{d}{dx}\left[(1+x)^n\right] &= \frac{d}{dx}\left[\sum_{k=0}^{n} \binom{n}{k} x^k\right] \\ n(1+x)^n &= \sum_{k=1}^{n} k\binom{n}{k} x^{k-1} \\ x=-1 \implies 0 &= \sum_{k=1}^{n} k\binom{n}{k}(-1)^{k-1}. \end{aligned}$$

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Brualdi 5.18.

Solution.

Method 1: Using the committee with chairperson idea, $\frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}\binom{n+1}{k+1}$, we have

$$\begin{aligned} \sum_{k=0}^{n}(-1)^k \frac{1}{k+1}\binom{n}{k} &= \sum_{k=0}^{n}(-1)^k \frac{1}{n+1}\binom{n+1}{k+1} \\ &= \frac{1}{n+1}\sum_{j=1}^{n+1}(-1)^{j-1}\binom{n+1}{j}. \end{aligned}$$

The sum would be 0 if we started at $j=0$. We are missing one term, so we have

$$\begin{aligned} \frac{1}{n+1}\sum_{j=1}^{n+1}(-1)^{j-1}\binom{n+1}{j} &= \frac{1}{n+1}\binom{n+1}{0} \\ &= \frac{1}{n+1}. \end{aligned}$$

There is another method of solution (possibly easier) which is as follows:

Method 2: Using the binomial theorem, we have

$$\begin{aligned}
(1+x)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k \\
\int_0^t (1+x)^n dt &= \int_0^t \sum_{k=0}^{n} \binom{n}{k} x^k dt \\
\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^t &= \left[\sum_{k=0}^{n} \binom{n}{k} x^{k+1} \frac{1}{k+1}\right]_0^t \\
\frac{(1+t)^{n+1}}{n+1} - \frac{1}{n+1} &= \sum_{k=0}^{n} \binom{n}{k} t^{k+1} \frac{1}{k+1} \\
t=-1 \implies -\frac{1}{n+1} &= \sum_{k=0}^{n} \frac{(-1)^{k+1}}{k+1} \binom{n}{k} \\
\frac{1}{n+1} &= \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{n}{k}.
\end{aligned}$$

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Brualdi 5.19.

Solution.

$$\begin{aligned}
\sum_{m=1}^{n} m^2 = \sum_{m=1}^{n} \left[2\binom{m}{2} + \binom{m}{1}\right] &= 2\sum_{m=1}^{n} \binom{m}{2} + \sum_{m=1}^{n} \binom{m}{1} \\
&= 2\binom{n+1}{3} + \binom{n+1}{2} \\
&= 2 \cdot \frac{(n+1)n(n-1)}{6} + \frac{(n+1)n}{2} \\
&= \frac{n(n+1)}{6}\left[2(n-1)+3\right] = \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

■

Brualdi 5.25.

Solution.

Both sides count the number of n-subsets of an (m_1+m_2)-set. Equation (5.16) is the case where $m_1 = m_2 = n$. ■

Brualdi 5.33.

Solution.

The inductive partitioning of an n-set X into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ chains gives us a systematic way of creating symmetric chain decompositions (this was the second proof of Sperner's Theorem). The decomposition for $X = \{1, 2, 3, 4\}$ is given in the book, so we use the algorithm to produce the $\binom{5}{2} = 10$ chains for $X = \{1, 2, 3, 4, 5\}$.

$$
\begin{array}{l}
\emptyset \subset \{1\} \subset \{1,2\} \subset \{1,2,3\} \subset \{1,2,3,4\} \subset \{1,2,3,4,5\} \\
\qquad \{5\} \subset \{1,5\} \subset \{1,2,5\} \subset \{1,2,3,5\} \\
\qquad \{4\} \subset \{1,4\} \subset \{1,2,4\} \subset \{1,2,4,5\} \\
\qquad\qquad \{4,5\} \subset \{1,4,5\} \\
\qquad \{2\} \subset \{2,3\} \subset \{2,3,4\} \subset \{2,3,4,5\} \\
\qquad\qquad \{2,5\} \subset \{2,3,5\} \\
\qquad\qquad \{2,4\} \subset \{2,4,5\} \\
\qquad \{3\} \subset \{1,3\} \subset \{1,3,4\} \subset \{1,3,4,5\} \\
\qquad\qquad \{3,5\} \subset \{1,3,5\} \\
\qquad\qquad \{3,4\} \subset \{3,4,5\}
\end{array}
$$

■

Brualdi 5.40.

Solution.

The coefficient of $x_1^3 x_2^3 x_3 x_4^2$ is given by the term $\binom{9}{3,3,1,2} x_1^3 (-x_2)^3 (2x_3)(-2x_4)^2$ so the coefficient is $-8\binom{9}{3,3,1,2}$. ■

Extra Problem #1.

Solution.

The fraction (percentage) of level 5 that is used is $\frac{28}{\binom{8}{5}} = \frac{28}{56} = \frac{1}{2}$ and the fraction (percentage) of level 3 is $\frac{30}{\binom{8}{3}} = \frac{30}{56}$, which means $\sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} > 1$. By LYM, this is impossible.

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Extra Problem #2.

Solution.

The number of chains of each length is given in the following chart:

Length	Number
13	$\binom{12}{0}$
11	$\binom{12}{1} - \binom{12}{0}$
9	$\binom{12}{2} - \binom{12}{1}$
7	$\binom{12}{3} - \binom{12}{2}$
5	$\binom{12}{4} - \binom{12}{3}$
3	$\binom{12}{5} - \binom{12}{4}$
1	$\binom{12}{6} - \binom{12}{5}$
Even	0

$\blacksquare$