Name: ____________________

Math 378 Spring 2011
Assignment 2
Solutions

Brualdi 2.1.

The number with neither property is 5^4.

The number with property (a) is $P(5,4) = 5!$.

The number with property (b) is $5^3 \cdot 2$.

The number with property (a) and (b): two ways to pick the last digit, then $4 \cdot 3 \cdot 2$ for the rest, so 48.

Brualdi 2.7.

There are 3! circular permutations for the 4 men to sit leaving two spaces between each two men, then there are 8! ways to seat the women in the 8 empty seats. This means there are $6 \cdot 8!$ ways. If the seats are labelled then there are 12 times as many ways (ie. $12 \cdot 6 \cdot 8!$ ways).

Brualdi 2.9.

There are 14! circular 15-permutations. There are 13! ways to have AB somewhere and 13! ways to have BA somewhere. This means there are $14! - 2 \cdot 13!$ ways if B and A refuse to sit next to each other, but $14! - 13!$ ways if B ony refuses to sit on A's right.

Brualdi 2.13.

(a) There are two different ways to see this.

View #1: If you label the students A,B, or C depending on their dorm assignment, each way of assigning the dorms corresponds to a permutation of $\{25 \cdot A, 35 \cdot B, 40 \cdot C\}$, so there are $\dfrac{100!}{25!35!40!}$ ways.

View #2: There are $\binom{100}{25}$ ways to choose who goes in A, then $\binom{75}{35}$ ways to choose who goes in B, and everyone else goes into C. This gives $\binom{100}{25}\binom{75}{35}$ ways.

(b) There are $\binom{50}{25}$ ways to fill A, $\binom{50}{35}$ ways to fill B, so $\binom{50}{25}\binom{50}{35}$ ways.

Brualdi 2.19.

(a) There are 8! ways to choose the spots occupied by the rooks and $\binom{8}{5}$ ways to choose which are red, so $8!\binom{8}{5}$ ways.

(b) There are $\binom{12}{8}$ ways to choose the 8 rows the rooks will occupy, then $\binom{12}{8}$ ways to choose the columns. After that, we have an 8x8 board, so we have $8!\binom{8}{5}$ ways to place them. This results in $\binom{12}{8}\binom{12}{8}\cdot 8!\binom{8}{5}$ ways total.

Brualdi 2.20.

There are 9! ways total. If 0 and 9 are opposite, there are 8! ways to fill in the rest. So there are $9! - 8!$ ways.

Brualdi 2.28.

(a) The easiest way to think about this is as a permutation of $\{9 \cdot E, 8 \cdot N\}$. There are $\binom{17}{8}$ total paths.

(b) There are $\binom{7}{3}$ ways to get to the west end of the underwater block and $\binom{9}{4}$ ways to get from the east end of the underwater block to work, so there are $\binom{17}{8} - \binom{7}{3}\binom{9}{4}$ paths that avoid the underwater block.

Brualdi 2.31.

There are $P(15, 3)$ ways to assign gold, silver, and bronze and there are $\binom{12}{3}$ ways to drop three of the twelve remaining teams. This means there are $P(15, 3)\binom{12}{3}$ total possible outcomes for the tournament.

Brualdi 2.32.

The book's answer counts the number of 11-permutations of the 3 possible 11-sets you get by removing one symbol.

Notice that the number of 11-permutations of the 12-set is the same as the number of 12-permutations (I'll leave it up to you to think about why, but I mentioned this idea in class). This is $\frac{12!}{3!4!5!}$.

Brualdi 2.35.

There are 6 different 3-combinations:

$$\{\{2 \cdot a, 1 \cdot b\}, \{2 \cdot a, 1 \cdot c\}, \{1 \cdot a, 2 \cdot c\}, \{1 \cdot a, 1 \cdot b, 1 \cdot c\}, \{1 \cdot b, 2 \cdot c\}, \{3 \cdot c\}\}$$

Brualdi 2.36.

You can take $0, 1, 2, \ldots, n_i$ objects of type i, so $n_i + 1$ possibilities for each $i = 1, 2, \ldots, k$. Thus, there are $(n_1 + 1)(n_2 + 1) \cdots (n_k + 1) = \prod_{i=1}^{k} (n_i + 1)$ possible combinations of any size.

Brualdi 2.38.

Let $y_1 = x_1 - 2$, $y_2 = x_2$, $y_3 = x_3 + 5$, $y_4 = x_4 - 8$. Then we want the number of solutions to

$$\begin{aligned} (y_1 + 2) + (y_2) + (y_3 - 5) + (y_4 + 8) &= 30 \\ y_1 + y_2 + y_3 + y_4 &= 25 \end{aligned}$$

where the y_i's are nonnegative integers. By the divider idea we know there are $\binom{28}{3}$.

Brualdi 2.45.

(a) If we only care about the number of books on the shelves, then the books can be considered to be identical. In other words, this will be the number of solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ where the x_i's are nonnegative integers. This is $\binom{24}{4}$.

(b) If we now have distinct books, but don't care about the ordering on the shelves, we can think of this as trying to create 5 sets X_1, X_2, X_3, X_4, and X_5 that represent the shelves. We take each book and have 5 choices of where to put it. This is 5^{20}.

(c) This is still different than the first two, although you can slightly modify 1 to work. The easiest way to think about this is to start with 24 blanks. Pick the spots where the shelf dividers will go (like the balls/barriers idea). Then in the remaining 20 blanks, we can place the distinct books. Since the order on each shelf matters, we get $\binom{20}{4}20!$.

Brualdi 2.51.

Choose <u>any</u> subset of $\{1, 2, \ldots, n\}$, then add a's until the set is size n. Thus there are 2^n.

Brualdi 2.57.

We choose the rank of the pair, then the two suits of the pair, then the remaining 3 cards of different ranks and their suits. This results in $13\binom{4}{2}\binom{12}{3}4^3$ possible hands with exactly one pair.

Alternatively, we could: pick the 4 ranks, pick the rank to pair, then choose the suits of the pair and the suits of the non paired cards which is $\binom{13}{4} \cdot 4 \cdot \binom{4}{2} \cdot 4^3$.

This means the probability of getting exactly one pair is $\dfrac{13\binom{4}{2}\binom{12}{3}4^3}{\binom{52}{5}} \approx .422569$.

Brualdi 2.63.

(a) This is the same as the number of solutions to $x_1+x_2+x_3+x_4 = 6$ in positive integers, or $y_1 + y_2 + y_3 + y_4 = 2$ in nonnegative integers. This is $\binom{5}{3} = 10$, so the probability is $\frac{10}{6^4}$.

(b) The number of ways to have no 1s is 5^4; precisely one 1, $4 \cdot 5^3$; precisely two 1s, $\binom{4}{2}5^2$. The total of these is $5^4 + 4 \cdot 5^3 + 6 \cdot 5^2 = 5^2(25 + 20 + 6) = 5^2(51) = 1275$, so the probability is $\frac{1275}{6^4} = \frac{1275}{1296}$.

(c) There cannot be any ones, so there are only 5 choices for each die. This means the probability is $\frac{5^4}{6^4}$.

(d) There are $6 \cdot 5 \cdot 4 \cdot 3$ ways, so the probability is $\frac{6 \cdot 5 \cdot 4 \cdot 3}{6^4} = \frac{5}{18}$.

(e) There could be three of one kind (6 choices) and one of the other (5 choices) with 4 ways to choose the color of the die with the second number, so $6 \cdot 5 \cdot 4$. There also could be two of one and two of another, so $\binom{6}{2} = 15$ ways to choose the two doubled numbers. Then we have $\binom{4}{2} = 6$ ways to choose the two colors of dice for one of the numbers, so $15 \cdot 6$ of these. This means there are $120 + 90 = 210$ ways total, so the probability is $\frac{35}{216}$.

Extra Problem 1.

(a) This is $\left(\frac{4}{5}\right)^{10}$.

(b) The number of solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 10$ in nonnegative integers is $\binom{14}{4}$ and in positive integers (ie. $y_1 + y_2 + y_3 + y_4 + y_5 = 5$) is $\binom{9}{4}$.

This means the probability is $\frac{\binom{14}{4} - \binom{9}{4}}{\binom{14}{4}}$.

Extra Problem 2.

(a) This is $\binom{30}{3}$.

(b) We now (somewhat carefully) count the number of "bad" ways. First, there are 28 ways to have 3 consecutive. Now we count the ways of having precisely 2 consecutive (ie. the third is nonconsecutive). If we think of the pair as $i, i+1$, then there are 26 choices for the third integer if $i = 2, 3, \ldots, 28$. If $i = 1$ or $i = 29$, then there are 27 choices for the third integer. So the number of "bad" ways is $28 + 27 \cdot 26 + 2 \cdot 27 = 28 + 28 \cdot 27 = 28^2$. So the total number of "good" ways would be $\binom{30}{3} - 28^2$.