# EXAM #2 Solutions 2012-3-5

1. Consider the statement "If $n$ is a prime and $n > 2$, then $n$ is odd" where $n \in \mathbb{N}$.

   (a) (8 points) Write the negation of the statement. Is the negation of the statement true or false? Provide a counterexample or some short explanation.

   > **Solution:** "$n$ is prime and $n > 2$ and $n$ is even."
   > This statement is false since any even $n > 2$ has a factor of 2 and so is not prime. Another possibility is that if any prime larger than 2 can be written as $2n$ for some integer $n$ (ie. is not even), then it would have a factor of 2 (contradicting the fact that it was prime).

   (b) (8 points) Write the contrapositive of the statement. Is the contrapositive of the statement true or false? Provide a counterexample or some short explanation.

   > **Solution:** "If $n$ is even, then $n$ is not prime or $n \leq 2$."
   > This statement is true. Note that for all even $n \in \mathbb{N}$, either $n = 2$ or $n > 2$. For $n = 2$, the statement holds (since $n \leq 2$). For even $n > 2$, since $n$ is even, $n = 2k$ for some integer $k$. This means $n$ has a factor of 2 which means $n$ is not prime.

2. (12 points) Prove that the square of any odd integer $j$ can be written in the form $8k + 1$ for some $k \in \mathbb{Z}$. In other words: "If $j$ is an odd integer, there exists an integer $k$ such that $j^2 = 8k + 1$."

> **Solution:** Let $j$ be an odd integer. This means there exists a value $x \in \mathbb{Z}$ such that $j = 2x + 1$. Consider $j^2$.
>
> $$j^2 = (2x + 1)^2 = 4x^2 + 4x + 1 = 4x(x + 1) + 1.$$
>
> Note that $x$ is either even or odd. Thus, $x(x + 1)/2 = k$ is an integer and so we can write $j^2 = 8k + 1$ where $k \in \mathbb{Z}$.

3. (8 points) Let $n$ be an integer. Use contraposition to prove that if $n$ is not even, then $n + 1$ is not odd.

> **Solution:** The contrapositive statement is "if $n + 1$ is odd, then $n$ is even." Let $n + 1$ be odd. This means that $n + 1 = 2k + 1$ for some $k \in \mathbb{Z}$. Subtracting 1 from both sides, we have $n = 2k$ which, since $k \in \mathbb{Z}$, shows $n$ is even.

4. Express each of the following sums using summation notation.

   (a) (4 points) $4+5+6+7$

   **Solution:** One possibility is $\sum_{i=4}^{7} i$.

   (b) (5 points) $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{20}$

   **Solution:** One possibility is $\sum_{i=1}^{10} \frac{1}{2i}$.

   (c) (5 points) $5+9+13+17+21+\cdots+49$

   **Solution:** One possibility is $\sum_{i=0}^{11}(4i+5)$. Another is $\sum_{i=1}^{12}(4i+1)$.

5. (10 points) Prove that for $t \in \mathbb{N}$,

$$\left(\frac{t(t+1)}{2}\right)^2 + (t+1)^3 = \left(\frac{(t+1)\left[(t+1)+1\right]}{2}\right)^2.$$

(Hint: Do **not** use induction.)

**Solution:** Notice that

$$\begin{aligned}
\left(\frac{t(t+1)}{2}\right)^2 + (t+1)^3 &= \left(\frac{(t+1)^2}{2}\right)\left[t^2+4(t+1)\right] \\
&= \left(\frac{(t+1)^2}{2}\right)\left[t^2+4t+4\right] \\
&= \left(\frac{(t+1)^2}{2}\right)[t+2]^2 \\
&= \left(\frac{(t+1)^2}{2}\right)[(t+1)+1]^2 \\
&= \left(\frac{(t+1)\left[(t+1)+1\right]}{2}\right)^2.
\end{aligned}$$

6. (10 points) Prove that $\sum_{i=1}^{n}(2i-1) = n^2$ for all $n \in \mathbb{N}$.

**Solution:** There are 3 different methods for this:

***Method #1:*** **[By Induction]** Base Case: Notice that $\sum_{i=1}^{1}(2i-1) = 1 = 1^2$.

Inductive Step: Assume $\sum_{i=1}^{n-1}(2i-1) = (n-1)^2$. Consider $\sum_{i=1}^{n}(2i-1)$.

$$\begin{aligned}\sum_{i=1}^{n}(2i-1) &= \sum_{i=1}^{n-1}(2i-1) + (2n-1)\\ &= (n-1)^2 + (2n-1)\\ &= n^2 - 2n + 1 + (2n-1)\\ &= n^2.\end{aligned}$$

Thus, the inductive step holds and so $\displaystyle\sum_{i=1}^{n}(2i-1) = n^2$ for all $n \in \mathbb{N}$.

***Method #2:*** Let $S = \sum_{i=1}^{n}(2i-1)$. We list two copies of the sum, one above the other

$$\begin{array}{ccccccccc} 1 & + & 3 & + & 5 & + & \cdots & + & (2n-1)\\ (2n-1) & + & (2(n-1)-1) & + & (2(n-2)-1) & + & \cdots & + & 1 \end{array}$$

This shows that

$$\begin{aligned} 2S &= \underbrace{2n + \cdots + 2n}_{n \text{ terms}}\\ 2S &= n(2n)\\ 2S &= 2n^2\\ S &= n^2. \end{aligned}$$

***Method #3:*** Consider $2n^2$ points in a grid with $2n$ rows and $n$ columns. Counted by columns, there are $2n^2$ points. We can also group the points into two disjoint subsets with columns of sizes 1, 3, 5, …, $2n-1$.