# EXAM #1 Solutions 2012-02-06

1. Consider $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6\}$ and $C = \{1, 2\}$ as subsets of $[9] = \{1, 2, \ldots, 9\}$.
   (a) (2 points) Is $2 \in A$? Yes
   (b) (2 points) Is $2 \in B$? No
   (c) (3 points) Is $2 \subseteq A$? No
   (d) (3 points) Is $\{2\} \subseteq A$? Yes
   (e) (3 points) Is $\emptyset \in A$? No
   (f) (3 points) Is $\emptyset \subseteq A$? Yes
   (g) (4 points) Find $A \cup B$.

   **Solution:** $A \cup B = \{1, 2, 3, 4, 5, 6\}$

   (h) (4 points) Find $A \cap B$.

   **Solution:** $A \cap B = \{4, 5\}$

   (i) (4 points) Find $B \cap C$.

   **Solution:** $B \cap C = \emptyset$

   (j) (4 points) Find $A \setminus C$.

   **Solution:** $A \setminus C = \{3, 4, 5\}$

   (k) (4 points) Find $\mathcal{P}(C)$.

   **Solution:** $\mathcal{P}(C) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

   (l) (4 points) Is $A \subseteq B$? Why or why not.

   **Solution:** No, because $1, 2, 3 \in A$ but $1, 2, 3 \notin B$.

   (m) (4 points) Is $C \subseteq A$? Why or why not.

   **Solution:** Yes. All of the elements of $C$ are also elements of $A$.

   (n) (6 points) Let $A \triangle B = (A \cup B) \setminus (A \cap B)$. This is called the ***symmetric difference*** of $A$ and $B$. Find $A \triangle B$.

   **Solution:** $A \triangle B = \{1, 2, 3, 6\}$

2. (a) (3 points) Let $O$ be the set of all odd integers. Write, using set notation, the set of all odd integers.

**Solution:** The set of odd integers is $O = \{2k+1 \mid k \in \mathbb{Z}\}$.

(b) (8 points) Write, using set notation, **two different representations** of a set that contains the quotients of any two odd integers.

**Solution:** Two different representations of this set would be $\left\{\frac{m}{n} \;\middle|\; m, n \in O\right\}$ or $\left\{\frac{2k+1}{2j+1} \;\middle|\; j, k \in \mathbb{Z}\right\}$.

(c) (4 points) Is the set from part (b) equal to $\mathbb{Q}$? Why or why not.

**Solution:** The set from part (b) is not equal to $\mathbb{Q}$. One of the reasons why is because $\frac{1}{2}$ is not an element of the set from part (b).

How do we know? Suppose we could represent $\frac{1}{2}$ as a quotient of two odd numbers, this would mean $\frac{1}{2} = \frac{m}{n}$ where $m$ and $n$ are odd. However, since this reduces to $\frac{1}{2}$, we have that $n = 2m$. But this means that $n$ is even. $\lightning$. Thus, $\frac{1}{2}$ cannot be in the set from part (b). ■

3. (10 points) A fraternity has a rule for new members: each must always tell the truth or always lie. They know who does which. If I meet three of them on the street and they make the following statements, which ones (if any) should I believe? Explain your reasoning.

A says: "All three of us are liars."

B says: "Exactly two of us are liars."

C says: "The other two are liars."

**Solution:** Notice that $A$ must be a liar, since if we believe him, he contradicts himself. Since he is a liar this means that it must be false that all three of them are liars which means that either $B$ or $C$ are truth tellers (or both).

Now, $C$ claims "The other two are liars". If we believe him, then $B$ is telling the truth, which means $C$ must be a liar.

Now, we can see that $B$ is making a true statement and so he must be a truth teller.

Therefore, we should only believe $B$.

4. (10 points) Show that the logical expression $S$ is equivalent to the logical expression $\sim S \implies (R \wedge \sim R)$.

**Solution:**

| $S$ | $\Longleftrightarrow$ | $[\sim S$ | $\implies$ | $(R$ | $\wedge$ | $\sim R)$ |
|---|---|---|---|---|---|---|
| **T** | **T** | F | **T** | T | **F** | F |
| **T** | **T** | F | **T** | F | **F** | T |
| **F** | **T** | T | **F** | T | **F** | F |
| **F** | **T** | T | **F** | F | **F** | T |

5. Provide an argument or counterexample to support the truth value of the following statements.

(a) (5 points) $\exists x \in \mathbb{R}, x^2 - x = 0$.

**Solution:** True since the value $x = 0$ or $x = 1$ satisfy the equation.

(b) (5 points) $\forall x \in \mathbb{R}, \sqrt{x^2} = x$.

**Solution:** False, any $x < 0$ is a counterexample.

(c) (5 points) $\exists x, y \in \mathbb{R}, x + y + 3 = 8$.

**Solution:** True since $x = 0, y = 5$ satisfies the equation.

(d) (5 points) $\forall x, y \in \mathbb{R}, x + y + 3 = 8$

**Solution:** False since $x = y = 0 \in \mathbb{R}$ fails.