Name: ____________________________

Math 283 Spring 2012
Assignment 5 Solutions

Exercise 4.2.

Solution.

Let $x = 333_{(12)} = 3 \cdot 111_{(12)}$ and $y = 333_{(5)} = 3 \cdot 1111_{(5)}$. It suffices to compare $144+12+1$ and $125+25+5+1$. The first is larger (by 1), so $x > y$. ∎

Exercise 4.18.

Solution.

We use induction on n to prove this for the power $2n+1$ (an odd number).

Basis step ($n = 1$): Here exponentiation is the identity function, so $x < y$ does in fact give that $x^1 < y^1$.

Induction step: Suppose that exponentiation to the power $2n-1$ is strictly increasing. Thus if $x < y$, then

$$x^{2n-1} < y^{2n-1}. \qquad (\star)$$

If $0 \leq x < y$, then $0 < x^2 < y^2$ and multiplying equation $(\star)$ gives us that $x^{2n+1} < y^{2n+1}$.

If $x < 0 \leq y$, then x^{2n+1} is negative and y^{2n+1} is nonnegative, so $x^{2n+1} < y^{2n+1}$.

If $x < y \leq 0$, then $0 \leq -y < -x$, and we proved that $(-y)^{2n+1} < (-x)^{2n+1}$. Since an odd power of -1 is -1, this gives us that $-y^{2n+1} < -x^{2n+1}$, and thus we have $x^{2n+1} < y^{2n+1}$.

Solutions to $x^n = y^n$. All pairs with $x = y$ are solutions. When n is odd, the exponentiation is strictly increasing, and hence in this case there are no other solutions. When n is even, the solutions are $x = \pm y$. To show that there are no other solutions, it suffices to show that exponentiation to the nth power is injective from the set of positive real numbers to itself. This follows by induction almost exactly like that above. ∎

Exercise 4.20.

Solution.

a) As proved in class, the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (ax + by, cx + dy)$ is a bijection if and only if $ad - bc \neq 0$. In this problem, the values taken by a, b, c, d are $a, -b, b, a$, respectively, and hence $ad - bc$ becomes $a^2 + b^2$, which by hypothesis is non-zero. Hence the given function is a bijection.

b) When f is a bijection, the inverse function f^{-1} gives for each element of the target the unique element of the domain that maps to it. Computing the inverse function may allow us to prove injectivity and surjectivity simultaneously. In this example, the inverse image of the element (r, s) in the target is the set of solutions (x, y) to the system of equations

$$\begin{cases} r = ax - by \\ s = bx + ay \end{cases}$$

Because $a^2 + b^2 \neq 0$, there is a unique solution (existence implies surjectivity and uniqueness implies injectivity). The unique solution of the system is $x = \dfrac{ra + bs}{a^2 + b^2}$ and $y = \dfrac{-br + as}{a^2 + b^2}$. Hence the inverse function is $f^{-1}(r, s) = \left(\dfrac{ra + bs}{a^2 + b^2}, \dfrac{-br + as}{a^2 + b^2}\right)$. ■

Exercise 4.21.

Solution.

Let A be the collection of even subsets of $[n]$, and let B be the collection of odd subsets. For each $X \in A$, define $f(X)$ as follows:

$$f(X) = \begin{cases} X \setminus \{n\} & \text{if } n \in X \\ X \cup n & \text{if } n \notin X. \end{cases}$$

By this definition, $|X|$ and $|f(X)|$ differ by one, so $f(X)$ is a set of odd size and f maps A to B.

We claim that this is a bijection. Consider distinct $X, Y \in A$. If both contain or both omit n, then $f(X)$ and $f(Y)$ agree on whether they contain n, but (since they were distinct) differ outside of $\{n\}$. If exactly one of either X or Y contains n, then exactly on of $f(X)$ or $f(Y)$ contains n. Thus $X \neq Y$ implies $f(X) \neq f(Y)$ and so f is injective.

If $Z \in B$, then reversing whether n is present in Z yields a subset X such that $f(X) = Z$. This means f is also surjective. Therefore, f is a bijection and so $|A| = |B|$.

Alternatively, we could define $g : B \to A$ by the same rule used to define f (swithcing the domain and target), and observe that $g \circ f$ is the identity function on A and $f \circ g$ is the identity function on B. This implies that g is the inverse of f and thus that f is a bijection and therefore $|A| = |B|$. Without knowing $|A| = |B|$ beforehand, it does not suffice to show that one of the compositions is the identity (we must show both). ■

Exercise 4.29 (extra problem).

Solution.

(a) The functions f and g are not injective, but h is injective. Since $g(x) = g(-x)$ for all x, g is not injective. Also, since $f(2) = f(1/2)$, among others, f is not injective.

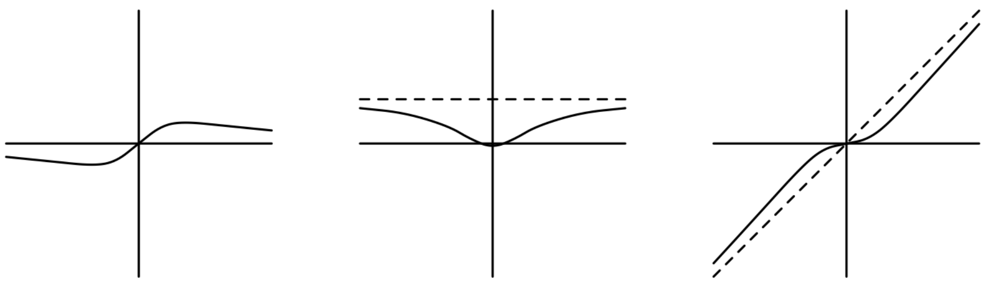
For h , we set $h(x) = h(y)$ and assume that $x \neq y$. We find that $x^3 + x^3y^2 = y^3 + x^2y^3$, which reduces to $x^2 + xy + y^2 = -x^2y^2$ after rearranging and then we divide by $x - y$. Rewriting this as $x^2(1 + y^2) + yx + y^2 = 0$ yields a quadratic equation for x in terms of y.

Since $b^2 - 4ac = y^2 - y^2 r(1+y^2) < 0$, there is no solution for x. Hence there are no distinct x and y with $h(x) = h(y)$.

(b) The functions f and g are not surjective. For all x, $g(x) > 0$. (Furthermore, $\frac{x^2}{1+x^2} = \frac{1}{1/x^2+1} < 1$ for all $x \neq 0$, so we always have $0 \leq g(x) < 1$.)

Also, $f(x) < 1$ for all x. If $x < 0$, then $f(x) < 0$. If $0 \leq x < 1$, then $x/(1+x^2) < x < 1$. If $x \geq 1$, then $x/(1+x^2) = 1/(1+1/x) < 1$.

(c) Note that $f(-x) = -f(x), g(-x) = g(x)$, and $h(-x) = -h(x)$. All are 0 at 0. Also, they are asymptotic to 0, 1, and x, respectively.



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Exercise 4.33 (extra problem).

Solution.

(a) Assume that f and g are injective. Suppose that $(g \circ f)(x) = (g \circ f)(x')$, ie $g(f(x)) = g(f(x'))$. Since g is injective this means $f(x) = f(x')$. Since f is injective, this means $x = x'$. Hence $g \circ f$ is injective.

Alternatively, consider the contrapositive. For $x, x' \in A$ with $x \neq x'$, we have $f(x) \neq f(x')$ because f is injective. Then $g(f(x)) \neq g(f(x'))$ because g is injective. Thus $x \neq x'$ implies $(g \circ f)(x) \neq (g \circ f)(x')$, so $g \circ f$ is injective.

(b) Assume that f and g are surjective. Let z be an arbitrary element of C. Since g is surjective, there is an element $y \in B$ such that $g(y) = z$. Since f is surjective, there is an element $x \in A$ such that $f(x) = y$. Hence we have found an element of A, namely x, such that $(g \circ f)(x) = z$ and so $g \circ f$ is surjective.

(c) By (a) and (b), $g \circ f$ is both injective and surjective, hence it is a bijection.

(d) By part (c), $g \circ f$ is a bijection from A to C. Thus $g \circ f$ is invertible, and the inverse is defined to be the function that yields the identity function on A when composed with $g \circ f$. Let I_A and I_B denote the identity functions on A and B. Letting $h = f^{-1} \circ g^{-1}$, we use the associativity of composition to obtain $h \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ I_B \circ f = f^{-1} \circ f = I_A$. Thus h is the inverse of $g \circ f$.

Alternatively, we could argue that $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ have the same domain and target and have the same value at each element of the domain so they are the same function. ■

Exercise 4.37.

Solution.

Suppose that $f(x) = f(y)$. Because f is a function, we can apply it to this element to obtain $f(f(x)) = f(f(y))$. By the definition of composition, this yields $(f \circ f)(x) = (f \circ f)(y)$. The hypothesis that $f \circ f$ is injective now implies that $x = y$. We have now proved that $f(x) = f(y)$ implies $x = y$, and thus f is injective. ■

Exercise 4.40.

Solution.

We assert that $h^n = f^{-1} \circ g^n \circ f$.

To prove this, we use induction on n. For $n = 1$, it holds by the definition of h. For $n > 1$, we use the definition of nth iterate, the induction hypothesis and the associativity of composition to compute

$$\begin{aligned} h^n &= h \circ h^{n-1} = (f^{-1} \circ g \circ f) \circ (f^{-1} \circ g^{n-1} \circ f) \\ &= f^{-1} \circ g \circ (f \circ f^{-1}) \circ g^{n-1} \circ f = f^{-1} \circ g \circ g^{n-1} \circ f \\ &= f^{-1} \circ g^n \circ f. \end{aligned}$$

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Exercise 4.42.

Solution.

We use induction on n.

Basis step ($n = 0$): In this case, $[n] = \emptyset$, and a function from A to $\emptyset$ can be defined only if $A = \emptyset$. Hence $m = 0$.

Inductive step ($n > 0$): Let f be a bijection from $[m]$ to $[n]$. Let $r = f^{-1}(n)$. Define g by $g(k) = f(k)$ for $k < r$, while $g(k) = f(k+1)$ for $k \geq r$; this function maps $[m-1]$ into $[n-1]$. Since f is a bijection and we have used all images under f except $f(r)$, g is a bijection. By the induction hypothesis, $m - 1 = n - 1$, and hence $m = n$. ■

Exercise 4.43.

Solution.

There is a bijection from a set A to a proper subset B of A only if A is finite. If A is finite, then also B is finite. Let $m = |A|$ and $n = |B|$. By the definition of size, there are bijections $f : A \to [m]$ and $g : B \to [n]$. Let h be a bijection from A to B. Now $g \circ h \circ f^{-1}$ is a bijection from $[m]$ to $[n]$. By the previous exercise, $m = n$. This contradicts the hypothesis that B is a proper subset of A. Hence the hypothesis that A is finite must be false. ■

Exercise 4.45.

Solution.

If $f : A \to A$ and A is finite, then f is injective if and only if f is surjective.

We use the method of contradiction to prove each direction of the claim. First suppose that f is injective, but some $y \in A$ is not in the image of f. Each inverse image has size at most one (since f is injective) and $I_f(y)$ is empty. Hence the total is less than $|A|$. This is a contradiction because there are $|A|$ elements in the domain.

Now suppose that f is surjective, but $f(x) = f(x') = y$ for some distinct $x, x' \in A$. Each inverse image has size at least one (since f is surjective) and $I_f(y)$ has size at least 2. Hence the total is more than $|A|$. This is a contradiction, because the inverse images partition the domain, which has only $|A|$ elements.

If $A = \mathbb{N}$ and f is defined by $f(x) = 2x$, then f is injective but not surjective. Hence the claim does not hold when A is infinite. ■

Exercise 4.46.

Solution.

Suppose that A and B are finite and $f : A \to B$.

(a) If f is injective, then $|A| \le |B|$. Since f is injective, each element of B is the image of at most 1 element of A. When we sum the contribution 0 or 1 over all elements of B (depending on whether the element is in the image), we obtain $|A|$ (each element of A has an image in B) and the sum is at most $|B|$ (each element of B contributes at most once).

(b) If f is surjective then $|A| \ge |B|$. When f is surjective, each element of B belongs to the image of f. By the definition of a function, then inverse images of the elements of B are pairwise disjoint subsets of A. Therefore, picking one element from the inverse image of each element of B yields $|B|$ distinct elements of A. This is a subset of A, so $|B| \le |A|$.

(c) If A and B are finite and $f : A \to B$ and $g : B \to A$ are injections, then $|A| = |B|$ and f and g are bijections. Applying (a) to f yields $|A| \le |B|$. Applying (a) to g yields $|B| \le |A|$. Hence $|A| = |B|$. Since f is injective, its image has $|A|$ elements; since $|A| = |B|$, the image is all of B and f is surjective. By the same argument, g is surjective. Thus, f and g are bijections. ■

Exercise 4.47.

Solution.

Every even natural number is obtained by doubling a unique natural number, so doubling is a bijection from $\mathbb{N}$ to the set of even numbers. This means that the set of even natural numbers is countable.

The operation of adding 1 is a bijection from the set of odd natural numbers to the set of even natural numbers. This means that the set of odd natural numbers is countable. ■

Exercise 4.49.

Solution.

Let $\{A_i : i \in \mathbb{N}\}$ be the sets, and let B be their union. Since each A_i is countable, for each i there is a sequence $\{a_{i,j} : j \in \mathbb{N}\}$ listing the elements of A_i once and only once. View these elements as listed at the points (j, i) in the first quadrant of the Cartesian plane with the elements of A_i in the ith row (this picture is similar to the one we used in class).

To show that B is countable, it suffices to construct a sequence listing each element of B once and only once. Each element of B now appears at a point in the first quadrant, but it appears more than once if it belongs to more than one of the sets. The positions with $i + j - 1 = k$ form the kth diagonal of the arrangement; every element appears in some diagonal. We form the sequence by listing the elements of the first diagonal, then the second, and so on in increasing order of k as in the bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. Within each diagonal, we use increasing order in j. However, whenever we encounter an element that already appears in our list, we skip it to avoid listing elements more than once. Since each diagonal is finite, we eventually reach each specified diagonal and thus each specified element. ∎

Extra Problem #1.

Solution.

The paper referenced provides the key ideas in the following geometric arguments to reduce the problem to the case $x^3 + rx$.

The formula for the value of a general cubic polynomial at x is $f(x) = ax^3 + bx^2 + cx + d$ where the coefficients a, b, c and d are given with $a \neq 0$. Since multiplying the function by -1 doesn't affect injectivity (one-to-one), we may assume that $a > 0$.

We assert that every cubic polynomial has rotational symmetry about a point. This is similar to the idea of and odd polynomial, except the rotational symmetry is about a point (m, n) instead of $(0, 0)$. The function $f(x) = ax^3 + bx^2 + cx + d$ has rotational symmetry about a point (m, n) if and only if a function g given by $g(x) = f(x + m) - n$ is an odd function. Notice that this transformation moves the point (m, n) on the graph of f to the origin.

Note that a cubic function is odd if and only if

$$\begin{aligned} ax^3 + bx^2 + cx + d &= -\left[a(-x)^3 + b(-x)^2 + c(-x) + d\right] \\ &= ax^3 - bx^2 + cx - d. \end{aligned}$$

which is true if and only if $b = d = 0$. Further, note that $n = f(m) = am^3 + bm^2 + cm + d$. Thus, we can rewrite g as follows:

$$\begin{aligned} g(x) &= f(x + m) - n \\ &= a(x + m)^3 + b(x + m)^2 + c(x + m) + d - (am^3 + bm^2 + cm + d) \\ &= ax^3 + 3ax^2m + 3axm^2 + bx^2 + 2bxm + cx \\ &= ax^3 + (3am + b)x^2 + (3am^2 + 2bm + c)x. \end{aligned}$$

Recall, that the coefficient of x^2 must be 0, which means

$$m = -\frac{b}{3a}.$$

Thus we know that f has rotational symmetry about the point

$$\left(-\frac{b}{3a}, f\left(-\frac{b}{3a}\right)\right).$$

Thus, we can rewrite g as follows:

$$\begin{aligned} g(x) &= ax^3 + (3am^2 + 2bm + c)x \\ &= ax^3 + \left(3a\left(-\frac{b}{3a}\right) + 2b\left(-\frac{b}{3a}\right) + c\right)x \\ &= ax^3 + \left(3a\left(\frac{b^2}{(3a)(3a)}\right) - \frac{2b^2}{3a} + \frac{3ac}{3a}\right) \\ &= ax^3 + \left(\frac{3ac - b^2}{3a}\right). \end{aligned}$$

Now, dividing by a does not affect the injectivity of a function and so we can consider the function h defined by $g(x)/a$. In other words, $h(x) = x^3 + rx$ where $r = \dfrac{3am^2 + 2bm + c}{a}$.

We can now proceed two different ways.

Method #1: First, we realize that for the function to be injective, the maximum and minimum value must be the same. The paper then provides a method of determining the coordinates of the maximum and minimum value which we then would set equal to each other for the condition.

Method #2: Notice that if $x^3 + rx = (x')^3 + rx'$ for some distinct x and x', then dividing by $x - x'$ yields $x^2 + xx' + (x')^2 = -r$.

If r is negative, then $(x, x') = (0, \sqrt{-r})$ is a solution, and so the function is not injective. If r is 0, then there is no solution with $x \neq x'$ (since cubing is injective). If r is positive, then again, there is no solution, because $x^2 + yx' + (x')^2$ is never negative (since $a^2 + b^2 \geq 2\,|a|\,|b|$).

Thus, h is injective if and only if $r \geq 0$ and this determines whether f is injective. This means that $3ac - b^2 \geq 0$.

Therefore, the requirement for injectivity of a general cubic function $b^2 - 3ac \leq 0$. ■

Extra Problem #2.

Solution.

(a) Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $f(m, n) = 2m + n$. The function is not an injection since $(0, 2)$ and $(1, 0)$ both map to 2. The function is surjective. Let $a \in \mathbb{Z}$. Then the element $(0, a)$ maps to a satisfying the definition of surjectivity.

(b) Let $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $g(m, n) = 6m + 3n$. The function is not an injection since $(0, 2)$ and $(1, 0)$ both map to 6. The function is also not surjective. Note that

$6m+3n = 3(2m+n)$ and so all of the outputs will be multiples of 3. Thus 2 is not a possible output of g. ■

Extra Problem #3.

Solution.

Since $A \setminus B \subseteq A$, we know that $A \setminus B$ is countable. Now let us assume that $A \setminus B$ is finite, in other words, let $|A \setminus B| = n$. Note that $(A \setminus B) \cup B$ is the union of disjoint sets. In class we proved that if the sets were disjoint, then the cardinality of their union was the sum of their cardinalities. Thus $|(A \setminus B) \cup B| = m + |B|$. But $m + |B| \in \mathbb{N}$ since B is finite. This means that A is finite. $\lightning$. Thus, $A \setminus B$ must be countably infinite. ■

Extra Problem #4.

Solution.

Let S be a countable set and assume that $A \subseteq S$. There are two cases: A is finite or A is infinite.

If A is finite, it is countable.

If A is infinite, let $f : S \to K \subseteq \mathbb{N}$ be a bijection (which exists since S is countable). Now define $g : A \to f(A)$ by $g(x) = f(x)$ for every $x \in A$. We assert that g is a bijection on A.

Suppose $a \neq b$ where $a, b \in A$. Then $g(a) \neq g(b)$ since $g(a) = f(a)$, $g(b) = f(b)$, and $f(a) \neq f(b)$.

By definition of g it is onto (since that's how we defined it!).

Thus we have a bijection from A onto $f(A) \subseteq K \subseteq \mathbb{N}$ ■