Math 283 Spring 2012
Assignment 4 Solutions

D'Angelo & West 3.22.

Solution.

We use induction on n.

When $n = 1$, the two sides are equal. When $n = 2$, the statement is the ordinary triangle inequality (Proposition 1.3).

For the induction step, suppose that the inequality holds when $n = k$; this is the induction hypothesis. We prove that if $k \geq 2$, then the inequality also holds when $n = k + 1$, using the ordinary triangle inequality and the induction hypothesis applied to the first k numbers. We compute

$$\left|\sum_{i=1}^{k+1} a_i\right| = \left|a_{k+1} + \sum_{i=1}^{k} a_i\right| \leq |a_{k+1}| + \left|\sum_{i=1}^{k} a_i\right| \leq |a_{k+1}| + \sum_{i=1}^{k+1} |a_i| = \sum_{i=1}^{k+1} |a_i|.$$

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D'Angelo & West 3.26.

Solution.

We use induction on n.

Basis step: $a_1 = 1 = 1^3 - 1 + 1$.

Induction step: Given that $a_k = k^3 - k + 1$, we have

$$a_{k+1} = a_k + 3k(k+1) = k^3 - k + 1 + 3k^2 + 3k = (k+1)^3 - k = (k+1)^3 - (k+1) + 1.$$

■

D'Angelo & West 3.33.

Solution.

The number of closed subintervals with integer endpoints contained in the interval $[1, n]$ (including one-point intervals) is $\frac{n(n+1)}{2}$. This is because there are $n - i$ subintervals of length i, for $1 \leq i \leq n - 1$.Thus the total count is the sum of the integers from 1 (when $i = n - 1$) to n (when $i = 0$). ■

D'Angelo & West 3.35.

Solution.

When $n = 1$, the formula reduces to 1, which is $\sum_{i=1}^{0} q^i$, To prove the formula for a positive integer $n = k$ assuming it holds when $n = k - 1$, we have

$$\sum_{i=0}^{k} q^i = q^k + \sum_{i=0}^{k-1} q^i = q^k + \frac{q^k - 1}{q - 1} = \frac{q^{k+1} - q^k + q^k - 1}{q - 1} = \frac{q^{k+1} - 1}{q - 1}.$$

■

D'Angelo & West 3.37.

Solution.

Based on #35, we have $\sum_{i=1}^{n} n^i = \sum_{i=0}^{n} n^i - 1 = \frac{n^{n+1} - 1}{n - 1} - 1 = \frac{n^{n+1} - n}{n - 1}$. ■

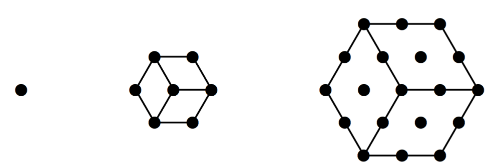
D'Angelo & West 3.39.

Solution.

Let a_n be the number of dots in the hexagonal array S_n with n rings. We use summation formulas for the first m integers and the first m squares to compute a_n and $\sum_{k=0}^{n} a_k$. As illustrated, $a_1 = 1$. Beyond that, ring i adds $6(i - 1)$ dots, so $a_n = 1 + \sum_{i=2}^{n} 6(i - 1) = 1 + 6\sum_{i=1}^{n-1} i = 1 + 3n(n - 1)$ for $n \geq 1$. Furthermore,

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (1 - 3k + 3k^2) = n - 3\frac{n(n+1)}{2} + 3\frac{n(n+1)(2n+1)}{6}.$$

After some algebraic effort, this simplifies to n^3. The answer n^3 can be explained directly by viewing S_n as the "front" of a cubical array of dots viewed from the vertex of 3 sides as illustrated below. ■



D'Angelo & West 3.43.

Solution.

With $x = 1, y = 1$, we have

$$\begin{aligned} f(1 \cdot 1) &= 1 \cdot f(1) + 1 \cdot f(1) \\ f(1) &= f(1) + f(1) \\ 0 &= f(1). \end{aligned}$$

For the second statement, we use induction on n.

Basis step: For $n = 1$, we have $f(x^1) = f(x) + xf(1) = f(x) = 1x^0 f(x) = nx^{n-1}f(x)$.

Induction step: For $n > 1$, we use the induction hypothesis for $n - 1$ to compute

$$\begin{aligned} f(x^n) &= f(x \cdot x^{n-1}) \\ &= xf(x^{n-1}) + x^{n-1}f(x) \\ &= x(n-1)x^{n-2}f(x) + x^{n-1}f(x) \\ &= (n-1)x^{n-1}f(x) + x^{n-1}f(x) \\ &= nx^{n-1}f(x). \end{aligned}$$

■

D'Angelo & West 3.46.

Solution.

For $n = 1$, the condition is $x + x < x^2$. When x is positive, this is equivalent to $x > 2$. Thus, the condition $x > 2$ is necessary. We can now either proceed by induction or do a direct proof. Both are shown below.

Method #1: [Induction on n]. **Basis step (n=1):** done above.

Inductive step: Suppose that $x^{n+1} > x^n + x$. Since $x > 2$, we have $x^2 > x$. Thus

$$x^{n+2} = x(x^{n+1}) > x(x^n + x) = x^{n+1} + x^2 > x^{n+1} + x.$$

Method #2: (Direct proof for all $n \in \mathbb{N}$). Since $x > 2$, we have $1/x^{n-1} \leq 1$, and thus $1 + 1/x^{n-1} \leq 2 < x$. Since $x > 0$, we can multiply both sides by x^n to obtain $x^n + x < x^{n+1}$. ■

D'Angelo & West 3.53.

Solution.

Basis step: When $n = 0$, f is a constant function, and we are given $c = f(0)$, so f is defined by $f(x) = c$.

Induction step: Suppose that $n \geq 1$. Given a polynomial f such that $f(n) = c$, let g be the polynomial defined by $g(x) = f(x) - c$. Since $g(n) = 0$, Theorem 3.24 gives us that

$g(x) = (x - n)h(x)$, where h is a polynomial of degree $n - 1$. If we can determine h, then we can determine f by $f(x) = (x - n)h(x) + c$.

Notice that since h is of degree $n - 1$ we are almost where we can use the inductive hypothesis, however, we need to be able to compute the values $h(0), h(1), \ldots, h(n - 1)$ first to satisfy the conditions of the inductive hypothesis. Since $h(x) = \dfrac{g(x)}{x - n}$ when $x \neq n$, then we have $h(i) = \dfrac{f(i) - c}{i - n}$ for $i = 1, \ldots, n - 1$. Since we know these values of f, we can obtain the values of $h(0), \ldots, h(n - 1)$. This means the inductive hypothesis holds and so we can determine f. $\blacksquare$