Math 283 Spring 2012
Assignment 3 Solutions

D'Angelo & West 2.30.

Solution.
(a). This is a conditional statement: "If the letter side is a vowel, then the number side is odd." The statement is false only when there is a card with one side that is a vowel and the other side is even. The statement is true if this never happens. Thus, to check this statement, we must look at the other side of every card showing a vowel or an even number. ■

(b). This is a biconditional statement which requires both the statement in (a) and it's converse. The converse of the statement in (a) is "If the number side is odd, then the letter side is a vowel." To check the converse, we must look at the other side of every card that is showing an odd number or a consonant. To check the statement in (a), we must look at the other side of every card showing an even number or a vowel. Thus, we must look at the other side of every card to test statement (b). ■

D'Angelo & West 2.33.

Solution.
The hats are from a set of two red and three black hats. The third child will see the first two hats, the second child sees the first, and the first child sees none. If the first two were both red, then the third would know she wore black. Since she is silent, at least one of the first two is black. The second, knows this, thus if she saw red on the first, she would know she wore black. Since she is silent, the first child's hat must be black. Thus she names black. ■

D'Angelo & West 2.40.

Solution.
(a). Each domino covers precisely one black square and one white square, so a board that is covered exactly by dominoes has the same number of white squares as black squares. Removing the opposite corners leaves us 32 black squares but only 30 white squares so it cannot be covered exactly by dominoes. ■

(b).
Method #1: Notice that 60 squares remain (30 white and 30 black) and so 15 T-shapes must be used in total. Note, each T-shape covers an odd number of squares of each color. Since the sum of 15 odd numbers is always odd, any board covered exactly by 15 T-shapes must have an odd number of squares of each color. Our board has 30 squares of each color, so it cannot be covered by 15 T-shapes.

Method #2: Since the region has the same number of squares of each color, we can conclude that we must use the same number of each of the two types of tiles. Thus an even number of tiles must be used, however, this contradicts the total of 60 squares since 60 is not 4 times an even number. ■

D'Angelo & West 2.41.

Solution.

When a person has the wrong hat, the real owner of that hat also has the wrong hat. This means that $k = 1$ must be excluded.

For $k = 0$, this means that all the people have their own hats (which is, of course, possible). For $k \geq 2$, give hat i to the $i+1$ person for each i from 1 to $k-1$ and give hat k to person 1. Then give all other hats to their proper owners. This shows that it is possible for all values of $k \geq 0$ except $k = 1$. ■

D'Angelo & West 2.50.

Solution.

(a). $(A \cup B)^c = A^c \cap B^c$.

(For all of these, element chasing arguments are fine as well, but here is another argument based on the logical interpretation for this problem.) The left hand side denotes the set of everything that is not in A or B. This consists of everything that is outside of A and outside of B, which is precisely the set described by $A^c \cap B^c$.

Element Chasing: Let $x \in (A \cup B)^c$. This happens only when $x \notin A \cup B$ which is equivalent to saying both $x \notin A$ and $x \notin B$. But, $x \notin A$ means $x \in A^c$ and $x \notin B$ means $x \in B^c$. Since $x \in A^c$ and $x \in B^c$, we know $x \in A^c \cap B^c$. This shows that $(A \cup B)^c \subseteq A^c \cap B^c$.

Now, let $x \in A^c \cap B^c$. This means $x \in A^c$ and $x \in B^c$. In other words, $x \notin A$ and $x \notin B$. This would mean that $x \notin A \cup B$, which is to say that $x \in (A \cup B)^c$. This shows that $A^c \cap B^c \subseteq (A \cup B)^c$.

Since we have subset inclusion both ways, we now know $(A \cup B)^c = A^c \cap B^c$. ■

(b). $A \cap [(A \cap B)^c] = A \setminus B$.

We can also prove these using set equivalence arguments (utilizing the equivalences on pages 33 and 34).

Note that $A \cap [(A \cap B)^c] = A \cap (A^c \cup B^c) = \underbrace{(A \cap A^c)}_{\emptyset} \cup (A \cap B^c) = A \cap B^c = A \setminus B$. ■

(c). $A \cap [(A \cap B^c)^c] = A \cap B$.

We have $A \cap [(A \cap B^c)^c] = A \cap (A^c \cup B) = \underbrace{(A \cap A^c)}_{\emptyset} \cup (A \cap B) = A \cap B$. ■

(d). $(A \cup B) \cap A^c = B \setminus A$.

We notice $(A \cup B) \cap A = \underbrace{(A \cap A^c)}_{\emptyset} \cup (B \cap A^c) = B \setminus A$ ■

D'Angelo & West 2.51.

Solution.

(a). We show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. An element in the set on the left must belong to A or it must belong to both B and C. In the first case, since it belongs to A it is in the set on the right. In the second case, since it belongs to both B and C, it is in the set on the right. A similar argument shows that every element of the set on the right is in the set on the left. ■

(b). We show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. An element in the set on the left must belong to A and it must also belong to B or C. In the first case, it belongs to A and B; in the second, A and C. Thus it is in the set on the right. A similar argument shows that every element of the set on the right is in the set on the left. ■

D'Angelo & West 2.54.

Solution.

The desired configuration has an odd number of blacks in every circle, which can only be acquired using operation (a). Since (a) flips four tokens in each circle, it does not change the parity of the number of black tokens in any circle. Hence a configuration with an odd number of blacks in every circle can only be achieved from another such configuration. Since the initial configuration is not of this type, the desired configuration cannot be reached. ■

D'Angelo & West 3.5.

Solution.

True. For $n = 1$, $2 \cdot 1 + 1 = 3 = 1^2 + 2 \cdot 1$.

If $\sum_{k=1}^{n}(2k+1) = (n^2 + 2n)$, then we have

$$\sum_{k=1}^{n+1}(2k+1) = (n^2 + 2n) + (2(n+1)+1) = (n^2 + 2n + 1) + 2(n+1) = (n+1)^2 + 2(n+1).$$

■

D'Angelo & West 3.7.

Solution.

This inequality fails when $n = 5$ since it is equivalent to $0 < n^2 - 10n + 25 = (n-5)^2$. ■

D'Angelo & West 3.11.

Solution.

We use induction on n to prove this for $n \geq 0$.

Basis step: The empty set is the only set of 0 elements, and $\emptyset$ is the only subset of $\emptyset$, so the formula 2^0 is correct when $n = 0$.

Induction step: Suppose that the claim is true when $n = k$. Let S be a set of $k+1$ elements, and let x be an element of S. The subsets of S consist of those containing x and those not containing x. The subsets not containing x are subsets of $S \setminus \{x\}$; by the induction hypothesis, there are 2^k of these. The subsets containing x consist of x together with a subset of $S \setminus \{x\}$; again the induction hypothesis implies that there are 2^k . Thus altogether there are $2^k + 2^k = 2^{k+1}$ subsets of S. ■

D'Angelo & West 3.14.

Solution.

Induction could be used on these, but I will use some properties of finite sums to show a different manner of solution.

(a). $\sum_{i=1}^{n}(4i-1) = 4\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = 2n(n+1) - n = n(2n+1).$ ■

(b). $\sum_{i=0}^{n}(4i+1) = 4\sum_{i=0}^{n} i + \sum_{i=0}^{n} 1 = 2n(n+1) + (n+1) = (n+1)(2n+1).$ ■

(c). $-1+2-3+4-\cdots-(2n-1)+2n = \sum_{i=1}^{n}[-(2i-1)+2i] = \sum_{i=1}^{n} 1 = n.$ ■

(d). $1-3+5-7+\cdots+(4n-3)-(4n-1) = \sum_{i=1}^{n}[(4i-3)-(4i-1)] = \sum_{i=1}^{n}(-2) = -2n.$ ■

D'Angelo & West 3.16.

Solution.

We use induction on n.

Basis Step: For $n = 1$, we have $\sum_{i=1}^{1} i^3 = 1 = \left(\frac{1 \cdot 2}{2}\right)^2$.

Induction Step: Suppose the claim holds when n is k. Using the induction hypothesis after isolating the last term, we have

$$\begin{aligned}\sum_{i=1}^{k+1} i^3 = (k+1)^3 + \sum_{i=1}^{k} i^3 &= (k+1)^3 + \left(\frac{k(k+1)}{2}\right)^2 \\ &= \frac{(k+1)^2}{4}\left[4(k+1) + k^2\right] \\ &= \frac{(k+1)^2}{4}(k+2)^2.\end{aligned}$$

■