1. Find the mass and centroid of the first-octant region that is interior to \( x^2 + z^2 = 1 \) and \( y^2 + z^2 = 1 \) (with density \( \delta \equiv 1 \)).

**Solution**
Project the solid on the \( yz \)-plane. Then on the \( yz \)-plane, the region \( R \) is bounded by \( y = 0 \), \( z = 0 \) and \( y^2 + z^2 = 1 \). Thus,

\[
m = \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-x^2}} dx dy dz = \int_0^1 (1-z^2) dz = \frac{2}{3}
\]

\[
\bar{x} = \frac{3}{2} \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-x^2}} x dx dy dz = \frac{3}{2} \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{1-z^2}{2} dy dz
\]

\[
= \frac{3}{2} \int_0^1 \left( 1 - \frac{z^2}{2} \right) dz
\]

\[
= \frac{3}{4} \left[ \frac{y}{5} (5-2y^2) \sqrt{1-y^2} + \frac{3}{8} \sin^{-1} y \right]_0^1 = \frac{9\pi}{64}
\]

\[
\bar{y} = \frac{3}{2} \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-x^2}} y dy dx dz = \frac{9\pi}{64}
\]

\[
\bar{z} = \frac{3}{2} \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-x^2}} z dy dx dz = \frac{3}{2} \int_0^1 (1-z^2) dz = \frac{3}{8}.
\]

2. Find the volume of the region that lies inside both \( x^2 + y^2 + z^2 = 4 \) and \( x^2 + y^2 - 2x = 0 \).

**Solution**
One can compute the volume of the first quadrant of the solid (thus the solid lies between \( z = \sqrt{4-(x^2+y^2)} \), and \( z = 0 \)). The projection on the \( xy \)-plane is a region bounded by \((x-1)^2 + y^2 = 1\) and \( x = 0 \), or \( r = 2 \cos \theta \) with \( 0 \leq \theta \leq \pi/2 \) in polar. Therefore, the volume is, in cylindrical coordinates,

\[
V = 4 \int_0^{\pi/2} \int_0^{2 \cos \theta} \int_0^{\sqrt{4-r^2}} r dz dr d\theta = 4 \int_0^{\pi/2} \int_0^{2 \cos \theta} r \sqrt{4-r^2} dr d\theta
\]

\[
= 4 \int_0^{\pi/2} \left[ \frac{-1}{3} (4-r^2)^{3/2} \right]_0^{2 \cos \theta} d\theta = \frac{32}{3} \int_0^{\pi/2} (1 - \sin^3 \theta) d\theta
\]

\[
= \frac{32}{3} \int_0^{\pi/2} (1 - (1 - \cos^2 \theta) \sin \theta) d\theta = \frac{32}{3} \left[ \theta + \cos \theta - \frac{\cos^3 \theta}{3} \right]_0^{\pi/2} = \frac{16}{9} (3\pi - 4).
\]

3. Find the volume of the region bounded by \( z = x^2 + 2y^2 \) and \( z = 12 - 2x^2 - y^2 \).

**Solution**
Project the intersection of the two surfaces down to the \( xy \)-plane (by getting rid of \( z \) in the system of equations \( z = x^2 + 2y^2 \) and \( z = 12 - 2x^2 - y^2 \)), we get a region \( R \) on the \( xy \)-plane bounded by \( x^2 + y^2 = 4 \). Use cylindrical coordinates, we have

\[
V = \int_{-\pi}^{\pi} \int_0^2 \int_{x^2+2y^2}^{12-2x^2-y^2} rdz dr d\theta
\]
\[
\int_{-\pi}^{\pi} \int_{0}^{2} r(12 - 3r^2)dr \, d\theta = \int_{-\pi}^{\pi} 12 \, d\theta = 24\pi.
\]

4. (Use spherical coordinates) Find the volume of the region bounded by the plane \( z = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

**Solution**

As at lateral the boundary of the cone, \( \tan \phi = 1 \), \( \phi \leq \frac{\pi}{4} \). Thus \( -\pi \leq \theta \leq \pi \), \( 0 \leq \phi \leq \pi/4 \) and \( 0 \leq \rho \leq \rho \sin \phi \).

\[
V = \int_{-\pi}^{\pi} \int_{0}^{\pi/4} \int_{0}^{\sec \phi} \rho^2 \sin \phi d\rho d\phi d\theta = \frac{1}{3} \int_{-\pi}^{\pi} \int_{0}^{\pi/4} \sec^3 \phi \sin \phi d\rho d\phi d\theta
\]
\[
= \frac{1}{3} \int_{-\pi}^{\pi} \int_{0}^{\pi/4} \sec^2 \phi \tan \phi d\phi d\theta \quad \text{set} \ u = \tan \phi
\]
\[
= \frac{1}{3} \int_{-\pi}^{\pi} d\theta \int_{0}^{1} u du = \frac{\pi}{3}.
\]

5. (Use cylindrical coordinates) Find the volume of the region bounded by the plane \( z = 1 \) and the cone \( z = \sqrt{x^2 + y^2} \).

**Solution**

For each fixed \( \theta \) with \( -\pi \leq \theta \leq \pi \), the cross section is a triangle on \( rz \)-plane bounded by \( r = z \), \( r = 0 \) and \( z = 1 \). Thus

\[
V = \int_{-\pi}^{\pi} \int_{0}^{1} \int_{0}^{z} rdrdzd\theta
\]
\[
= \int_{-\pi}^{\pi} \int_{0}^{1} \frac{z^2}{2} dzd\theta = \frac{2\pi}{6} = \frac{\pi}{3}.
\]