1. (10 %) Find the integral (clearly indicates what coordinate system is used, and the bounds of the integration) that computes the volume of of a solid that is bounded by $z = x^2 + y^2$, $x + y = 1$, $x = 0$, $y = 0$ and $z = 0$.

**Solution:** The top surface is $z = x^2 + y^2$ and the bottom is $z = 0$, over the region $R$ on the $xy$-plane bounded by $x + y = 1$, $x = 0$, $y = 0$. Therefore, using rectangular coordinates, the integral is

$$\text{Vol} = \int_0^1 \int_0^{1-x} \int_0^{x^2+y^2} dz dy dx.$$  

2. (10 %) Find the integral (clearly indicates what coordinate system is used, and the bounds of the integration) that computes the volume of of a solid that lies inside both $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$.

**Solution:** Let us compute only the upper half solid. The top surface is $z = \sqrt{4 - (x^2 + y^2)}$ and the bottom is $z = 0$, over the region $R$ on the $xy$-plane bounded by $x^2 + y^2 = 1$. Therefore, using cylindrical coordinates, the integral is

$$\text{Vol} = \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} r dz dr d\theta.$$  

3. (10 %) Find the integral (clearly indicates what coordinate system is used, and the bounds of the integration) that computes the volume of of a solid that lies inside both $x^2 + y^2 + z^2 = 16$ and $z = \sqrt{x^2 + y^2}$.

**Solution:** The top surface is $z = \sqrt{16 - (x^2 + y^2)}$ and the bottom is $z = \sqrt{x^2 + y^2}$. Therefore, it is more convenient to use spherical coordinates. The largest $\phi$ is when $z = \sqrt{x^2 + y^2} = r$, which means $\tan \phi = \frac{r}{z} = 1$, and so $\phi = \frac{\pi}{4}$. In this case $-\pi \leq \theta \leq \pi$, $0 \leq \phi \leq \pi/4$ and $0 \leq \rho \leq 4$. Therefore, the integral is

$$\text{Vol} = \int_{-\pi}^{\pi} \int_0^{\pi/4} \int_0^4 \rho^2 \sin \phi d\rho d\phi d\theta.$$  

One can also use cylindrical coordinates. For each fixed $\theta$ in the interval $[-\pi, \pi]$, the cross section can be viewed as a region $R$ in the $rz$-plane, where $R$ is bounded by $r = 0$ on the left, $z = r$ below and $z = \sqrt{16 - r^2}$ above. The maximum value of $r$ can be found by solving $z = r$ and $z = \sqrt{16 - r^2}$ for $r$. As $r^2 = z^2 = 16 - r^2$, we have $r = \sqrt{8}$ (note that $r > 0$ and so we throw away $r = -\sqrt{8}$ in the solution). Therefore,

$$\text{Vol} = \int_{-\pi}^{\pi} \int_0^{\sqrt{8}} \int_r^{\sqrt{16-r^2}} r dz dr d\theta.$$
4. (10 %) Compute the following integral by using the spherical coordinates.

\[ I = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z^2 \sqrt{x^2 + y^2 + z^2} \, dz \, dy \, dx. \]

**Solution:** The upper surface of the integration solid is \( z = \sqrt{4 - x^2 - y^2} \) and the bottom is \( z = 0 \). The projection of this solid on the \( xy \)-plane is the region enclosed by \( x^2 + y^2 = 4 \). In spherical coordinates, for each fixed \( \theta \) in \([-\pi, \pi]\), the cross section (the intersection of the plane \( \theta = \theta_0 \) and the solid) will be a quarter of the circle \( \rho^2 = 4 \) (with \( \theta \) being a constant). Therefore, \( 0 \leq \rho \leq 2 \) and \( 0 \leq \phi \leq \pi/2 \). Thus the integral is

\[ I = \int_{-\pi}^{\pi} \int_{0}^{\pi/2} \int_{0}^{2} \rho^2 \cos^2 \phi \sin \phi \, d\rho \, d\phi \, d\theta. \]

\[ = \frac{32}{3} \int_{-\pi}^{\pi} \left[ -\cos^3 \phi \right]_{0}^{\pi/2} \, d\theta = \frac{64\pi}{9}. \]

5. (10 %) Given a vector field \( \mathbf{F} = (3x, -2y, -4z) \), compute \( \text{div}\mathbf{F} \) and \( \text{curl}\mathbf{F} \).

**Solution:**

\[ \text{div}\mathbf{F} = 3 + (-2) + (-4) = -3. \]

\[ \text{curl}\mathbf{F} = (0 - 0, 0 - 0, 0 - 0) = (0, 0, 0) \]

6. (15 %) Given a function \( f(x, y) = xy \), and a curve \( C : x = 3t, y = t^4 \) with \( 0 \leq t \leq 1 \), find \( \int_C f \, ds \) and \( \int_C f \, dx \).

**Solution:** \( x' = 3 \) and \( y' = 4t^3 \). Thus \( ds = \sqrt{9 + 16t^6} \, dt \).

\[ \int_C f \, ds = \int_{0}^{1} 3t^5 \sqrt{9 + 16t^6} \, dt \quad \text{set} \ u = 9 + 16t^6 \]

\[ = \frac{1}{32} \int_{9}^{25} u^{1/2} \, du = \frac{1}{32} \left[ \frac{2u^{3/2}}{3} \right]_{9}^{25} = \frac{49}{24} \]

\[ \int_C f \, dx = \int_{0}^{1} 9t^5 \, dt = \frac{9}{6} = \frac{3}{2}. \]

7. (12 %) Given a vector field \( \mathbf{F} = (2xy^2 + 3x^2, 2x^2y + 4y^3) \), do the following.

(7A) Verify that this is a conservative field.
(7B) Find a potential of $\mathbf{F}$.

**Solution:** (7A) Here $P = 2xy^2 + 3x^2$ and $Q = 2x^2y + 4y^3$. As $P_y = 4xy = Q_x$, this is a conservative field.

(7B) Compute $f(x, y) = \int P \, dx = \int (2xy^2 + 3x^2) \, dx = x^2y^2 + x^3 + c(y)$. Then $2x^2y + 4y^3 = Q = f_y = 2x^2y + c'(y)$, and so $c'(y) = 4y^3$. Therefore $c(y) = \int 4y^3 \, dy = y^4$ and so $f(x, y) = x^2y^2 + x^3 + y^4$.

8. (12 %) Do both of the following.

(8A) Verify that $\mathbf{F} = (\cos y, -x \sin y)$ is a conservative field.

(8B) Compute the integral $\int_C \cos y \, dx - x \sin y \, dy$ for a curve $C$ from $(0, 0)$ to $(2, \pi)$.

**Solution:** (8A) Here $P = \cos y$ and $Q = -x \sin y$. As $P_y = -\sin y = Q_x$, this is a conservative field.

(8B) One solution is to find a potential function $f = x \cos y$, using a method similar to (7B). Therefore,

$$\int_C \cos y \, dx - x \sin y \, dy = f(2, \pi) - (0, 0) = 2(-1) - 0 = -2.$$

Another solution is to choose a specific path such as $C_1 : 0 \leq x \leq 2$ with $y = 0$ followed by $C_2 : 0 \leq y \leq \pi$ with $x = 2$. In this case, $dy = 0$ in $C_1$ and $dx = 0$ in $C_2$. As $\cos 0 = 1$ and $\int \sin y \, dy = -\cos y + C$, we have

$$\int_C \cos y \, dx - x \sin y \, dy = \int_{C_1} \cos 0 \, dx - \int_{C_2} 2 \sin y \, dy = \int_0^2 dx - 2 \int_0^\pi \sin y \, dy = -2.$$

9. (10 %) Find a potential function for the conservative field $\mathbf{F} = (yz, xz + y, xy + 1)$.

**Solution:** Let $C$ denote the straight line from $(0, 0, 0)$ to $(x_0, y_0, z_0)$. Then $C : x = x_0t, y = y_0t, z = z_0t$ with $0 \leq t \leq 1$. Let $f$ be a potential function of $\mathbf{F}$ such that $f(0, 0, 0) = 0$. Then

$$f(x_0, y_0, z_0) = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_0^1 (3x_0y_0z_0t^2 + y_0^2t + z_0) \, dt = x_0y_0z_0 + \frac{y_0^2}{2} + z_0.$$

Thus $f(x, y, z) = xyz + \frac{y^2}{2} + z$. 

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