1. (10 %) Let $\rho = 2 \sin \phi$ denote an equation in spherical coordinates.

(A) Convert it to cylindrical coordinates.
(B) Convert it to rectangular coordinates.

**Solution** Note that $r = \rho \sin \phi$. Then multiply $\rho$ to both sides to get $\rho^2 = 2 \rho \sin \phi$.

(A) For cylindrical coordinates, note that $\rho^2 = r^2 + z^2$. Thus the answer is

$$r^2 + z^2 = 2r.$$ 

(B) Apply $r^2 = x^2 + y^2$ to get the answer for rectangular coordinates

$$x^2 + y^2 + z^2 = 2\sqrt{x^2 + y^2} \text{ or } (x^2 + y^2 + z^2)^2 = 4x^2 + y^2.$$ 

2. (10 %) Evaluate the following limits

(A) $\lim_{(x,y) \to (0,0)} \frac{x^2 + 2 + y^2}{x^2 - 2 + y^2}$.

**Solution**

$$\lim_{(x,y) \to (0,0)} \frac{x^2 + 2 + y^2}{x^2 - 2 + y^2} = \frac{0^2 + 2 + 0^2}{0^2 - 2 + 0^2} = -1.$$ 

(B) $\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{x^2 - y^2}$.

**Solution** Let the limit be taken along the $x$-axis that is, set $y = 0$, we have $\lim_{(x,0) \to (0,0)} \frac{x^2 + 0^2}{x^2 - 0^2} = \frac{1}{1} = 1$; and let the limit be taken along the $y$-axis that is, set $x = 0$, we have $\lim_{(0,y) \to (0,0)} \frac{0^2 + y^2}{0^2 - y^2} = -1$. Therefore, the limit does not exist.

3. (10 %) Find an equation of the tangent plane at the point (1,-1,-1) to the surface $z = xy$.

**Solution** Compute $z_x = y$ and $z_y = x$. Therefore, a normal vector for the tangent plane at $(1,-1,-1)$ is $n = (-1,1,-1)$, and so the equation is

$$-(x - 1) + (y + 1) - (z + 1) = 0.$$ 

4. (15 %)

(A) Compute all the first order partial derivatives of $f(x,y,z) = (x^2 + y^3 + z^4)e^{xyz}$.

**Solution** Use product rule for each of the partial derivatives:

$$f_x = 2xe^{xyz} + (x^2 + y^3 + z^4)(yz)e^{xyz},$$
$$f_y = 3y^2e^{xyz} + (x^2 + y^3 + z^4)(xz)e^{xyz},$$
$$f_z = 4z^3e^{xyz} + (x^2 + y^3 + z^4)(xy)e^{xyz}.$$
(4B) Verify that $z_{xy} = z_{yx}$, where $z = x^2 e^{y^2}$.

**Solution** First compute $z_x = 2xe^{y^2}$ and $z_y = 2yx^2 e^{y^2}$. Then compute $z_{xy} = 4xye^{y^2}$ and $z_{yx} = 4ye^{y^2}$, and so $z_{xy} = z_{yx}$.

5. (10 % for correct procedure and 5% for accuracy of solution) Find the highest and the lowest point of the surface given by

$$z = f(x, y) = x^2 + 2xy + 3y^2$$

over a square region with vertices $(-1, -1), (-1, 1), (1, -1)$ and $(1, 1)$.

**Solution** We first compute $z_x = 2x + 2y$ and $z_y = 2x + 6y$. Setting $z_x = 0$ and $z_y = 0$ to get the only critical point $(0, 0)$. Note that $f(0, 0) = 0$.

Consider each of the boundaries. Let $L_1$ denote the boundary $\{(x, 1) : -1 \leq x \leq 1\}$. Then $f(x, 1) = x^2 + 2x + 3 = (x + 1)^2 + 2$. Therefore, apply Calculus I or High school algebra to get maximum $f(1, 1) = 6$ and minimum $f(-1, 1) = 2$.

Let $L_2$ denote the boundary $\{(x, -1) : -1 \leq x \leq 1\}$. Then $f(x, -1) = x^2 - 2x + 3 = (x - 1)^2 + 2$. Therefore, apply Calculus I or High school algebra to get maximum $f(-1, -1) = 6$ and minimum $f(1, -1) = 2$.

Let $L_3$ denote the boundary $\{(1, y) : -1 \leq y \leq 1\}$. Then $f(1, y) = 1 + 2y + 3y^2$. Therefore, apply Calculus I to get maximum $f(1, 1) = 6$ and minimum $f(-1, 1) = 2$.

Let $L_4$ denote the boundary $\{(-1, y) : -1 \leq y \leq 1\}$. Then $f(-1, y) = 1 - 2y + 3y^2$. Therefore, apply Calculus I to get maximum $f(-1, -1) = 6$ and minimum $f(-1, 1) = 2$.

Summing up, the highest points on the surface are $(1, 1, 6)$ and $(-1, -1, 6)$; and the lowest points is $(0, 0, 0)$.

6. (10 %) Find every point on the surface $z = 3x^2 + 12x + 4y^3 - 12y + 1$ at which the tangent plane is horizontal.

**Solution** Compute $z_x = 6x + 12$ and $z_y = 12y^2 - 12$. Therefore, the critical points are $(-2, 1)$ and $(-2, -1)$. Compute $f(-2, 1) = -19$ and $f(-2, -1) = -3$. Hence at $(-1, 1, -19)$ and at $(-1, -1, -3)$, the surface has horizontal tangent planes.

7. (10 %) Find the dimension of the open-topped (rectangular) box with volume 500 in$^3$ that has minimum total surface area.

**Solution** Let $x, y, z$ denote the dimension of the box. Then $xyz = 500$ or $z = \frac{500}{xy}$. The
total surface area is formulated as
\[ f(x, y) = xy + 2xz + 2yz = xy + 2(x + y) \]
\[ \frac{500}{xy} = xy + \frac{1000}{x} + \frac{1000}{y}. \]

Compute the partial derivatives to get
\[ f_x = y - \frac{1000}{x^2}, \quad f_y = x - \frac{1000}{y}. \]

Solve the system of \( f_x = 0 \) and \( f_y = 0 \) to get \( x = y = 10 \), and so \( z = \frac{500}{xy} = 5 \).

8. (20 %) Let \( r(t) = (e^t \cos t, e^t \sin t, e^t) \) be a space curve (viewed as a position vector of a moving particle). Compute each of the following.

(8A) The velocity, the speed and the unit tangent vector.
(8B) The acceleration.
(8C) The curvature at the point when \( t = 0 \).
(8D) The unit normal vector at the point when \( t = 0 \).

**Solution**

(8A) The velocity \( v = (e^t (\cos t - \sin t), e^t (\sin t + \cos t), e^t) \) and (use \( \sin^2 t + \cos^2 t = 1 \))
\[ v = \sqrt{(e^t (\cos t - \sin t))^2 + (e^t (\sin t + \cos t))^2 + e^{2t}} = e^t \sqrt{3}. \]

The unit tangent vector is
\[ T(t) = \frac{1}{e^t \sqrt{3}} (e^t (\cos t - \sin t), e^t (\sin t + \cos t), e^t) = \frac{1}{\sqrt{3}} (\cos t - \sin t, \sin t + \cos t, 1). \]

(8B) Differentiate \( v \) to get \( a \).
\[ a(t) = (e^t (\cos t - \sin t) + e^t (-\sin t - \cos t), e^t (\sin t + \cos t) + e^t (\cos t - \sin t), e^t) = e^t (-2 \sin t, 2 \cos t, 1). \]

(8C) At \( t = 0 \), \( v = (1, 1, 1), \ v = \sqrt{3}, \) and \( a = (0, 2, 1) \). Compute \( v \times a = (1, 1, 1) \times (0, 2, 1) = (-1, -1, 2) \), and so \( |v \times a| = \sqrt{1 + 1 + 4} = \sqrt{6} \). Then use it to compute the curvature
\[ \kappa(0) = \frac{|v \times a|}{|v|^3} = \frac{\sqrt{6}}{(\sqrt{3})^3} = \frac{\sqrt{2}}{3}. \]

(8D) At \( t = 0 \), \( T, a_N, \) and \( a_T \) are, respectively,
\[ T = \frac{1}{\sqrt{3}} (1, 1, 1), \ a_N = \kappa v^2 = \frac{\sqrt{2}}{3} (3) = \sqrt{2}, \ a_T = \frac{v \cdot a}{|v|} = \frac{2 + 1}{\sqrt{3}} = \sqrt{3}. \]

Thus
\[ N = \frac{1}{a_N} (a - a_T T) = \frac{1}{\sqrt{2}} (0, 2, 1) - \sqrt{3} \frac{1}{\sqrt{3}} (1, 1, 1) = \frac{1}{\sqrt{2}} (-1, 1, 0). \]