Extremal size of graphs without a nowhere-zero 3-flow

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Abstract
An asymptotically best possible bound for the size of a simple graph that does not admit a nowhere-zero 3-flow is determined.

1. Introduction
Graphs in this note are finite and may have loops and parallel edges. Groups in this note are finite Abelian groups. Throughout this note, $A$ denotes an Abelian (additive) group with 0 as the additive identity. For integer $n \geq 2$, $Z_n$ denotes the cyclic group of order $n$.

For a subset $X \subseteq E(G)$, the contraction $G/X$ is the graph obtained from $G$ by identifying the two ends of each edge in $X$ and then deleting the edges in $X$. Note that even when $G$ is simple, $G/X$ may have loops or multiple edges. For convenience, we write $G/e$ for $G/\{e\}$, where $e \in E(G)$. If $H$ is a subgraph of $G$, then $G/H$ denotes $G/E(H)$.

For a vertex $v \in V(G)$, let

$$E_D^-(v) = \{(u, v) \in E(G) : u \in V(G)\}, \text{ and } E_D^+(v) = \{(v, u) \in E(G) : u \in V(G)\}.$$ 

The subscript $D$ may be omitted when $D(G)$ is understood from the context. Let $E_G(v)$ denote the subset of edges incident with $v$ in $G$.

Fix an orientation $D$ of $G$. Let $A$ be a nontrivial Abelian group and let $A^*$ denote the set of nonzero elements in $A$. Define $F(G, A) = \{f :$
$E(G) \rightarrow A$ and $F^*(G, A) = \{f : E(G) \rightarrow A^*\}$. For each $f \in F(G, A)$, the boundary of $f$ is a function $\partial f : V(G) \rightarrow A$ defined by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in $A$. Unless otherwise stated, we shall adopt the following convenience: if $X \subseteq E(G)$ and if $f : X \rightarrow A$ is a function, then we regard $f$ as a function $f : E(G) \rightarrow A$ such that $f(e) = 0$ for all $e \in E(G) - X$. We also use that notation $(D, f)$ for a function $f \in F(G, A)$ to emphasize the orientation $D$.

Let $G$ be an undirected graph and $A$ be an Abelian group. Let $Z(G, A)$ denote the set of all functions $b : V(G) \rightarrow A$ such that $\sum_{v \in V(G)} b(v) = 0$. A graph $G$ is $A$-connected if $G$ has an orientation $D$ such that for every function $b \in Z(G, A)$, there is a function $f \in F^*(G, A)$ such that $b = \partial f$. For an Abelian group $A$, let $\langle A \rangle$ denote the family of graphs that are $A$-connected.

Suppose that $f \in F^*(G, A)$ with a fixed orientation $D$ such that some edge $e_0 = (u, v) \in E(G)$ is oriented from $u$ to $v$. If a new orientation $D'$ of $G$ is obtained by reversing the direction of $e_0$, then one can redefine $f'(e) = f(e)$ for $e \in E(G) - \{e_0\}$ and $f'(e_0) = -f(e_0)$. It is easy to see that $f' \in F^*(G, A)$ and $\partial f$ in $D$ equals $\partial f'$ in $D'$. Therefore, the property $G \in \langle A \rangle$ is independent of the orientation of $G$.

The concept of $A$-connectivity was introduced by Jaeger et al in [6], where nowhere-zero $A$-flows were successfully generalized to $A$-connectivities. A concept similar to the group connectivity was independently introduced in [7], with a different motivation from [6].

A nowhere-zero $A$-flow (abbreviated as $A$-NZF) in $G$ is a function $f \in F^*(G, A)$ such that $\partial f = 0$. When $A$ is the additive group of all integers, an $A$-NZF $f$ is a nowhere-zero $k$-flow (or just $k$-NZF) if $\forall e \in E(M), 0 < |f(e)| < k$. The following is well known.

(1.1) (Arrowsmith-Jaeger [1], Brylawski [3] and Tutte [11]) For a regular matroid $M$, $M$ has an $A$-NZF if and only if $M$ has an $|A|$-NZF.

The nowhere-zero-flow problems were introduced by Tutte [10]. (See Jaeger’s survey in [5] for more in the literature on nowhere-zero flows.) For an integer $k \geq 2$, $F_k$ denotes the collection of all graphs admitting a
nowhere zero $Z_k$-flow. By definition,

$$\langle Z_k \rangle \subseteq F_k.$$ (1)

The containment is in fact proper as can be seen in (2.2) below. Tutte [5] has several conjectures concerning what graph $G$ admits a $k$-NZF. Jaeger (see [5]) showed that every 2-edge-connected graph admits a 8-NZF. This result was improved by Seymour (see [5]) who proved that every 2-edge-connected graph admits a 6-NZF. Tutte (see [5]) conjectured that every 2-edge-connected graph admits a 5-NZF. It is well known that the Petersen graph $P_{10}$ does not have a 4-NZF, and when $n$ is odd, $W_n$, the wheel with $n + 1$ vertices does not have a 3-NZF.

It is natural to consider, for $k \in \{3, 4\}$, the existence of a function $f(k, n)$ such that every 2-edge-connected simple graph $G$ with $n$ vertices and with at least $f(k, n)$ edges must have a $k$-NZF. The following was proved in [8].

(1.2) ([8]) Let $G$ be a 2-edge-connected simple graph with $n \geq 18$ vertices. If

$$|E(G)| \geq \frac{(n - 17)(n - 18)}{2} + 34,$$

then either $G \in F_4$, or $G$ is contractible to the Petersen graph. The bound is asymptotically best possible.

The main result of this note is the following which determined the extremal size of simple 2-edge-connected graph which does not have a 3-NZF.

(1.3) Let $G$ be a 2-edge-connected simple graph with $n \geq 6$ vertices. If

$$|E(G)| \geq \binom{n - 5}{2} + 46,$$ (2)

then either $G \in F_3$ or $G$ can be contracted to a $K_4$.

The bound given in (1.3) is asymptotically best possible. Let $W_5$ denote the wheel which consists of a 5-cycle $z_1 z_2 z_3 z_4 z_5 z_1$ and a center $z$ together with the spoke edges $zz_i$ ($1 \leq i \leq 5$). Let $G(n)$ be a graph obtained from $W_5$ by replacing exactly one of its six vertices by a clique $K_{n-5}$, where $n \geq 8$ is an integer. Then since $G(n)$ can be contracted onto $W_5$ which does not have a nowhere zero 3-flow, $G(n) \not\in F_3$. Note that $n = |V(G(n))|$
and $|E(G(n))| = \frac{(n-5)(n-6)}{2} + 10$, and so

$$\lim_{m \to \infty} \frac{|E(G(n))|}{\frac{(n-5)(n-6)}{2} + 49} = 1.$$ 

2. Proof of the main result

The following results are known.

(2.1) (Proposition 3.2 of [9]) Let $H$ be a subgraph of $G$ and let $A$ be an Abelian group. Each of the following holds.

(i) If $H \in \langle A \rangle$ and if $e \in E(H)$, then $H/e \in \langle A \rangle$.

(ii) If $H \in \langle A \rangle$, then $G/H \in \langle A \rangle \iff G \in \langle A \rangle$.

(iii) If $H \in \langle Z_k \rangle$, then $G/H \in F_k \iff G \in F_k$.

(Catlin called nonempty graph families satisfying (i), (ii) and (iii) of (2.1) complete families. See [4].)

(2.2) (Lemma 3.3 of [9]) Let $n \geq 1$ be an integer and let $C_n$ denote the cycle of $n$ vertices. Then for an Abelian group $A$, $|A| \geq n + 1$.

(2.3) (Corollary 3.5 of [9]) Let $m \geq 5$ an integer. Then both $K_m \in \langle A \rangle$ and $K_m - e \in \langle A \rangle$ for any abelian group $A$ with $|A| \geq 3$.

(2.4) (Proposition 3.6 of [9]) Let $M_2$ be a matching of $K_5$. Then $K_5 - M_2 \in \langle A \rangle$ for any abelian group with $|A| \geq 3$.

(2.5) Every 2-edge-connected graph with at most 5 vertices is either in $F_3$ or is contractible to $K_4$.

Proof Since the 3-cycle is in $F_3$, we may assume that $|V(G)| \geq 4$ and that $G$ is simple. If $G$ has a vertex $v$ of degree 2, then by induction, we can pick $e' \in E_G(v)$ and conclude that either $G/e' \in F_3$, whence $G \in F_3$, or $G/e'$ is contractible to $K_4$, whence $G$ is contractible to $K_4$.

Thus we assume that $\delta(G) \geq 3$. If $|V(G)| = 4$, then $G = K_4$. Therefore we assume that $|V(G)| = 5$. By (2.3), we may assume that $G \neq K_5$. By $\delta(G) \geq 3$, either $G = K_5 - e$, for some $e \in E(K_5)$, or $G = K_5 - M_2$ for some maximal matching $M_2$ of $K_5$. It follows by (2.3) or (2.4) that $G \in \langle Z_3 \rangle \subseteq F_3$. □

Proof of (1.3) We argue by contradiction and assume that

$G$ is a counterexample with $|V(G)|$ minimized. 

(3) 

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(2.6) $G$ does not contain any nontrivial subgraph $H$ that is in $\langle Z_3 \rangle$.

**Proof:** Suppose not, and $G$ has a subgraph $H$ with $H \in \langle Z_3 \rangle$ and $|V(H)| \geq 2$. We may assume that $H$ is a maximal subgraph of $G$ that is in $\langle Z_3 \rangle$. By the maximality of $H$, $G/H$ is simple. If $H$ spans $G$, then by (2.1)(iii) and by (2.2), $G \in \langle Z_3 \rangle \subset F_3$, a contradiction. Hence we assume that $G/H$ is nontrivial also.

Let $n = |V(G)|$, $m = |V(H)|$ and $l = |V(G/H)| = n - m + 1$. Since $G$ is simple, $|E(H)| \leq m(m - 1)/2$, and so by (2), we have

$$|E(G/H)| = |E(G)| - |E(H)|$$

$$\geq \binom{n - 5}{2} + 46 - \binom{m}{2} = \frac{n^2 - 11n + 30 - m^2 + m}{2} + 46$$

$$= \frac{(l - 5)(l - 6)}{2} + 46 + (m - 1)n - m^2 - 4m + 5.$$

Thus $|E(G/H)| \geq (l - 5)(l - 6)/2 + 46$ if and only if $(m - 1)n - m^2 - 4m + 5 < 0$. Since $l = n - m + 1$, since $m \geq 2 > 1$ and since

$$(m - 1)n - m^2 - 4m + 5 = (m - 1)(n - m - 5) = (m - 1)(l - 6),$$

$(m - 1)n - m^2 - 4m + 5 < 0$ if and only if $l \leq 5$.

By (2.5), every 2-edge-connected graph with at most 5 vertices is either in $F_3$ or is contractible to $K_4$. Thus if $l \leq 5$, then either $G/H \in F_3$, whence by (2.1)(ii), $G \in F_3$; or $G$ is contractible to $K_4$, contrary to (3).

Hence we may assume that $l \geq 6$ and so by (4), $G/H$ satisfies that hypothesis of (1.3). Therefore by the minimality of $G$, $G/H$ is either in $F_3$, whence by (2.1)(ii), $G \in F_3$; or $G/H$ is contractible to $K_4$, whence $G$ is contractible to $K_4$, contrary to (3). This proves (2.6). □

**Proof of (1.3), continued** By (2.6) and by (2.3), $G$ does not have $K_5$ as a subgraph. However, by (2)

$$|E(G)| \geq \binom{n - 5}{2} + 46 \geq \frac{3n^2}{8}. \quad (5)$$

By Turán's Theorem ([2], page 109), $G$ must have a $K_5$, and so a contradiction obtains. This proves (1.3). □
References


