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Even subgraphs of a graph

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Abstract

In [Discrete Math. 101 (1992) 33 - 37], Fleischner proved that if \( G \) is a 2-edge-connected graph, then \( G \) has an even subgraph \( H \) with \( \delta(H) \geq 2 \) such that \( H \) contains all vertices of \( G \) with degree at least 3. In [J. Combinatorial Theory, Ser. B 35 (1983) 297 - 308], Bermond Jackson and Jaeger showed that every 2-edge-connected graph \( G \) has an even subgraph \( H \) with \( |E(H)| \geq \frac{2}{3}|E(G)| \). In this note, we shall show that if \( G \) is a 2-edge-connected graph, then each of the following holds:

(i) \( G \) has an even subgraph \( H \) such that \( H \) contains all vertices of degree at least 3 in \( G \) and such that \( H \) contains a given pair of adjacent edges in \( G \).

(ii) \( G \) has an even subgraph \( H \) such that \( H \) contains all vertices of degree at least 3 in \( G \) and such that \( |E(H)| \geq \frac{2}{3}|E(G)| \).

Graphs in this note are finite and undirected, and may have multiple edges and loops. For a graph \( G \), we denote \( O(G) \) the set of vertices of odd degree in \( G \). A graph \( G \) is even if \( O(G) = \emptyset \). Let \( e \) be an edge in \( G \). The contraction \( G/e \) is the graph obtained by identifying the two ends of \( e \) and by deleting the resulting edge.

For each integer \( i \geq 1 \), denote

\[
D_i(G) = \{ v \in V(G) : \delta_G(v) = i \} \quad \text{and} \quad D^*_i(G) = \bigcup_{j \geq i} D_j(G).
\]

Using the Splitting Lemma (Lemma III.26 of [5], see also [6]) and the Petersen 1-factor theorem, Fleischner in [4] proved the following
Theorem 1 (Fleischner, [4]) Let $G$ be a nontrivial graph without cut edges. Then $G$ has an even subgraph $H$ such that $\delta(H) \geq 2$ and such that $V(G) - D_2(G) \subseteq V(H)$.

In [1], Bermond, Jackson and Jaeger proved the following:

Theorem 2 (Bermond, Jackson, and Jaeger, [1]) Every 2-edge-connected graph $G$ has an even subgraph $H$ with $|E(H)| \geq \frac{2}{3}|E(G)|$.

The main purpose of this note is to present some extensions of these two theorems by showing the following Theorem 3. Our method is a modification of the arguments in both [1] and [4].

Theorem 3 Let $G$ be a 2-edge-connected graph. Then each of the following holds:

(i) $G$ has an even subgraph $H$ such that $H$ contains all vertices of degree at least 3 in $G$ and such that $H$ contains a given pair of adjacent edges in $G$.

(ii) $G$ has an even subgraph $H$ such that $H$ contains all vertices of degree at least 3 in $G$ and such that $|E(H)| \geq \frac{2}{3}|E(G)|$.

The following Theorem 4 is needed. The proof for Theorem 3 follows from Lemmas 5 and 6 below.

Theorem 4 (Edmonds, [3]) Let $G$ be a 2-edge-connected 3-regular graph. Then there is an integer $k \geq 1$ and a family of perfect matchings $(M_1, \ldots, M_{3k})$ such that each edge $e \in E(G)$ is in exactly $k$ of the $M_i$'s.

Let $v \in V(G)$. Define

$$E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}.$$ 

Lemma 5 Let $G$ be a 2-edge-connected graph. For any $v \in V(G)$ and for any two edges $e_1, e_2 \in E_G(v)$, $G$ has an even subgraph $H$ satisfying each of the following properties:

(i) $\delta(H) \geq 2$,

(ii) $D_2^*(G) \subseteq V(H)$, and

(iii) $\{e_1, e_2\} \subseteq E(H)$.

Proof We argue by contradiction. Let $G$ be a counterexample
such that

\[ \sum_{v \in D_2(G)} d_G(v) \text{ is minimized}, \quad (1) \]

and subject to (1),

\[ |E(G)| \text{ is minimized.} \quad (2) \]

We have the following observations.

Claim 1. \( \Delta(G) \leq 3 \) and so \( d_G(u) \leq 3 \).

- Suppose that \( u \in D_4^*(G) \). Let \( N_G(u) = \{u_1, \ldots, u_m\} \) where \( e_i = uu_i, 1 \leq i \leq 2 \). Let \( G' \) be the graph obtained from \( G \) by spli
ting \( u \) into two vertices \( u' \) and \( u'' \) such that \( u' \) is exactly adjacent to \( u_1, u_2 \) and \( u'' \), and such that \( u'' \) is exactly adjacent to \( u', u_3, \ldots u_m \). Note that if \( G' \) has a cut edge, then since \( G \) is 2-edge-connected, the cut edge in \( G' \) must be the new edge \( u'u'' \).

Case A1: \( u'u'' \) is a cut edge of \( G' \).

Let \( G'_1 \) and \( G'_2 \) be the two components of \( G' - u'u'' \) such that \( \{e_1, e_2\} \subseteq E(G'_1) \). Since \( G \) is 2-edge-connected, \( G'_1 \) and \( G'_2 \) are also 2-edge-connected. By (1) and (2), \( G'_1 \) has an even subgraph \( H_1 \) with \( \delta(H_1) \geq 2 \) and \( D_3^*(G'_1) \subseteq V(H_1) \), and \( \{e_1, e_2\} \subseteq E(H_1) \). Similarly, \( G'_2 \) also contains an even subgraph \( H_2 \) such that \( \delta(H_2) \geq 2 \) and \( D_3^*(G'_2) \subseteq V(H_2) \). Therefore \( H = G[E(H_1) \cup E(H_2)] \) is an even subgraph in \( G \) satisfying Lemma 5, contrary to the assumption that \( G \) is a counterexample.

Case A2: \( G' \) is 2-edge-connected.

By (1), \( G' \) has an even subgraph \( H' \) with \( \delta(H') \geq 2 \) and with \( D_3^*(G') \subseteq V(H') \) such that \( \{e_1, e_2\} \subseteq E(H') \).

Let \( H = H' \) if \( u'u'' \notin E(H') \) and \( H = H' \setminus \{u'u''\} \) if \( u'u'' \in E(H') \).

Then \( H \) will be the desired even subgraph in \( G \), contrary to the assumption that \( G \) is a counterexample. This proves Claim 1.

Since \( G \) is 2-edge-connected, by Claim 1, we have \( 2 \leq \Delta(G) \leq 3 \).

If \( G \) is 2-regular, then the theorem holds trivially. If \( G \) is a 3-regular, then let \( e_3 \) be the only edge in \( E_G(u) - \{e_1, e_2\} \). Since \( G \) is 2-edge-connected 3-regular graph, by Theorem 4, there is a perfect matching \( M \) of \( G \) such that \( e_3 \in M \). It follows the \( H = G - M \) is the desired even subgraph. A contradiction again.

Next we only need to consider that case that \( \Delta(G) = 3 \) and \( D_2(G) \neq \emptyset \). Suppose that \( G \) has a vertex \( w \in D_2(G) \).
Assume first that \( w \neq u \) and that \( E_G(w) = \{ e', e'' \} \). We may assume that \( e'' \notin \{ e_1, e_2 \} \), since \( w \neq u \). Then by (2), \( G/e'' \) has an even subgraph \( H'' \) with \( \delta(H'') \geq 2 \) and with \( D_3^*(G/e'') \subseteq V(H'') \) such that \( \{ e_1, e_2 \} \subseteq E(H'') \).

Let \( H = G[E(H'')] \) if \( e' \notin E(H'') \) and \( H = G[E(H'') \cup \{ e'' \}] \) if \( e' \in E(H') \). Then since \( w \in D_2(G) \), \( H \) will be the desired even subgraph in \( G \), contrary to the assumption that \( G \) is a counterexample.

Assume then \( w = u \in D_2(G) \). If \( G \) is spanned by an edge \( e_1 \), then the theorem holds trivially. Assume that is not the case, and so there is an edge \( e \in E(G) \setminus E_G(u) \) such that \( e \) and \( e_1 \) are adjacent in \( G \). By (2), \( G/e_1 \) has an even subgraph \( H_1 \) with \( \delta(H_1) \geq 2 \) and with \( D_3^*(G/e_1) \subseteq V(H_1) \) such that \( \{ e, e_2 \} \subseteq E(H_1) \). Thus by \( u \in D_2(G) \), \( G[E(H_1) \cup \{ e_1 \}] \) is a desired even subgraph, contrary to the assumption that \( G \) is a counterexample. This proves Lemma 5. \( \square \)

A graph \( G \) is a weighted graph if \( G \) is associated with a non-negative integer valued function \( w : E(G) \rightarrow \mathbb{Z}^+ \cup \{ 0 \} \), \( w \) is called the weight function). If \( X \subseteq E(G) \), then \( w(X) = \sum_{e \in X} w(e) \). If \( H \) is a subgraph, then \( w(H) = w(E(H)) \).

Lemma 6 Let \( G \) be a weighted graph with \( \kappa'(G) \geq 2 \) and with weight function \( w \). Then \( G \) has an even subgraph \( H \) with \( \delta(H) \geq 2 \) and with \( D_3^*(G) \subseteq V(H) \) such that \( w(H) \geq \frac{2}{3}w(G) \).

Proof As in the proof of Lemma 5, we argue by contradiction and assume that \( G \) is a counterexample such that

\[ \sum_{u \in D_2(G)} d_G(v) \text{ is minimized,} \]  
\[ \text{and subject to (3),} \] 
\[ |E(G)| \text{ is minimized.} \]  

If \( D_2(G) \neq \emptyset \), then let \( v \in D_2(G) \) and let \( E_G(v) = \{ e_1, e_2 \} \). Let \( G' \) denote the weighted graph obtained from \( G - v \) by adding a new edge \( e \) joining the two neighbors of \( v \) in \( G \), and by assigning the weight \( w(e) = w(e_1) + w(e_2) \). By (4), \( G' \) has an even subgraph \( H' \) with

\[ D_3^*(G') \subseteq V(H') \text{ and } w(H') \geq \frac{2}{3}w(G'). \]
Note that $D^*_3(G') = D^*_3(G)$ and $w(G') = w(G)$. It follows that

$$H = \begin{cases} 
G[E(H')] & \text{if } e \notin E(H') \\
G[E(H' - e) \cup \{e_1, e_2\}] & \text{otherwise}
\end{cases}$$

is the desired even subgraph. Hence we may assume that $\delta(G) \geq 3$.

Suppose that $u \in D^*_3(G)$. Let $N_G(u) = \{u_1, \cdots, u_m\}$ with $m \geq 4$. Let $e_i = uu_i$, $1 \leq i \leq 2$. Let $G''$ be the graph obtained from $G$ by splitting $u$ into two vertices $u'$ and $u''$ such that $u'$ is exactly adjacent to $u_1, u_2$ and $u''$, and such that $u''$ is exactly adjacent to $u', u_3, \cdots u_m$. Note that $G''$ may have $u'u''$ as an only cut edge since $G$ is 2-edge-connected. If this is the case, then interchange $u_2$ and $u_3$ can assume that the new graph $G''$ is 2-edge-connected. Let $e$ denote the new edge joining $u'$ and $u''$. Then one can view $E(G'') = E(G) \cup \{e\}$. Extend the domain of $w$ by defining $w(e) = 0$. Then $G''$ with the extended $w$ is a weighted graph. By (3), $G''$ has an even subgraph $H''$ such that

$$D^*_3(G'') \subseteq V(H'') \text{ and } w(H'') \geq \frac{2}{3}w(G'').$$

Note that $D^*_3(G) - \{u\} \subseteq D^*_3(G'')$ and $w(G) = w(G'')$. It follows that

$$H = \begin{cases} 
G[E(H')] & \text{if } e \notin E(H') \\
G[E(H'/e)] & \text{otherwise}
\end{cases}$$

is the desired even subgraph. Hence we may assume that $\delta(G) = 3$, and so $G$ is 3-regular.

When $G$ is 3-regular, Lemma 6 follows from Theorem 4. In fact, by Theorem 4, for some integer $k \geq 1$, $G$ has a family of perfect matchings $(M_1, \cdots, M_{3k})$ such that each edge $e \in E(G)$ is in exactly $k$ of the $M_i$'s.

Assume that $w(M_1) \leq w(M_2) \leq \cdots \leq w(M_{3k})$. Then $3kw(M_1) \leq \sum_{i=1}^{3k} w(M_i) = kw(E(G))$, and so $w(M_1) \leq \frac{1}{3}w(E(G))$. It follows that $H = G - M_1$ is an even subgraph with $\delta(H) \geq 2, D^*_3(G) \subseteq V(H)$ and $w(H) \geq \frac{2}{3}w(E(G))$. The proof of Lemma 6 is complete. □

**Proof of Theorem 3:** Theorem 3(i) follows from Lemma 5 and Theorem 3(ii) follows from Lemma 6 with $w(e) = 1$. □
References


