JACKSON'S CONJECTURE ON EULERIAN SUBGRAPHS

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Abstract

In [Discrete Math. 101 (1992), 351-360], Jackson conjectured that if G is a 2-edge-connected graph, then G has an eulerian subgraph H with \(|V(H)| \geq 2\) such that for each component F of G - V(H), there are at most three edges between F and H.

In [J. Graph Theory 10 (1986), 309-324], Thomassen conjectured that all 4-connected line graphs are hamiltonian.

In this note, we show the following: (1) Jackson's conjecture implies Thomassen's conjecture. (2) Jackson's conjecture holds for graphs having no pseudo-peripheral cuts of size 3. If a connected graph G does not have vertices of degree 3, and if L(G) is 4-connected, then L(G) is hamiltonian. (4) For any graph G, if L^2(G) is 4-connected, the L^2(G) is hamiltonian. (This result is best possible.)

1. Introduction

We follow the notation of [2] except otherwise noted. Graphs may have multiple edges but loops are prohibited. Let G be a graph. The set of all odd vertices of G is denoted by O(G). If O(G) = \emptyset, then G is an even graph. If G is even and connected, then G is eulerian. Note that K_1 is an eulerian graph. The line graph of G, written L(G), has E(G) as its vertex set, where two vertices are adjacent in L(G) if and only if the corresponding edges are adjacent in G. We denote L^2(G) = L(L(G)). Let D_1(G) denote the set of vertices of G of degree 1 (pendant vertices) in G.

In [8], Thomassen poses the following conjecture.

Conjecture 1 (Thomassen, [8]). If L(G) is 4-connected, then L(G) is hamiltonian.

The following theorem was proved by Zhan [10] and, independently, by Jackson [6].

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Theorem 1.1 (Zhan, [10]). If \( L(G) \) is 7-connected, then \( L(G) \) is Hamiltonian.

In fact, Zhan [10] proves a stronger result that if \( L(G) \) is 7-connected, then \( L(G) \) is Hamiltonian connected. Conjecture 1 also holds for planar graphs.

Theorem 1.2 (Lai, [7]). If \( G \) is a simple planar graph, then if \( L(G) \) is 4-connected, then \( L(G) \) is Hamiltonian.

In [1], Jackson has conjectured:

Conjecture 2 (Jackson, [1]). If \( G \) is a 2-edge-connected graph, then \( G \) has an Eulerian subgraph \( H \) with \( V(H) \neq \emptyset \) such that for each component \( F \) of \( G - V(H) \), there are at most three edges between \( F \) and \( H \).

We shall call an Eulerian subgraph \( H \) of \( G \) satisfying the conclusion of Conjecture 2 a J-subgraph of \( G \). A minimal edge cut \( \{X \subset E(G) \mid \text{all the edges in } X \text{ are incident with a single vertex } v \} \) such that one component of \( G - X \) is an edge-disjoint union of paths \( P_1, P_2, \ldots, P_m \) with \( V(P_i) \cap V(P_j) = \{v\} \), for any \( 1 \leq i < j \leq m \). Here we regard the single vertex graph \( K_1 \) as a path of length zero, and so the three edges incident with a vertex of degree 3 also form a peripheral edge cut of size 3. In this note, we shall prove the following results.

(a) Jackson's conjecture implies Thomassen's conjecture.
(b) Jackson's conjecture holds for graphs having no pseudo-peripheral cuts of size 3.
(c) As a corollary of (2), if a connected graph \( G \) does not have vertices of degree 3, and if \( L(G) \) is 4-connected, then \( L(G) \) is Hamiltonian.
(d) For any graph \( G \), if \( L^2(G) \) is 4-connected, then \( L^2(G) \) is Hamiltonian.

The result in part (d) is best possible in the following sense. Let \( P_{10} \) denote the Petersen graph and let \( SP_{10} \) denote the graph obtained from \( P_{10} \) by replacing each edge of \( P_{10} \) by a path of length 2. Then \( L^2(SP_{10}) \) is 3-connected, but \( L^2(SP_{10}) \) is not Hamiltonian. One can easily obtain an infinite family of such examples based on \( SP_{10} \).

2. Jackson's conjecture implies Thomassen's conjecture.

An Eulerian subgraph \( H \) of \( G \) is dominating if \( G - V(H) \) is edgeless. The following are needed.

Theorem 2.1 (Harary and Nash-Williams, [5]). Let \( G \) be a graph with \( |E(G)| \geq 3 \). Then \( L(G) \) is Hamiltonian if and only if \( G \) has a dominating Eulerian subgraph \( H \).

Lemma 2.2. Let \( G \) be a connected graph. If \( G \) has a J-subgraph, and if \( L(G) \) is 4-connected, then \( L(G) \) is Hamiltonian.
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Proof. By Theorem 2.1, it suffices to show that the J-subgraph \( H \) of \( G \) is a dominating eulerian subgraph of \( G \). If not, then there is an edge \( e \in E(G) \) lying in \( G - V(H) \). Let \( F \) be the component of \( G - V(H) \) containing \( e \). Then there is a set \( X \), say, of at most three edges between \( F \) and \( H \) in \( G \). Thus \( |X| \leq 3 \). By the definition of \( L(G) \), \( X \) is a vertex cut of \( L(G) \) contrary to the assumption that \( L(G) \) is 4-connected. Therefore, \( H \) must be a dominating eulerian subgraph of \( G \). This completes the proof.

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Theorem 2.3. Jackson’s conjecture implies Thomassen’s conjecture.

Proof. Let \( G \) be a graph with \( L(G) \) 4-connected, and suppose that Conjecture 2 is true. Let \( G_1 = G - D_1(G) \). Since \( L(G) \) is 4-connected, there is no edge cut \( X \subseteq E(G) \) of \( G \) with \( |X| \leq 3 \) such that both sides of \( G - X \) have edges. Therefore \( G - D_1(G) \) is 2-edge-connected.

By the truth of Conjecture 2, \( G_1 \) has a J-subgraph \( H \). Note that \( H \) is also an eulerian subgraph of \( G \). It suffices, by Lemma 2.2, to show that for every component \( F' \) of \( G - V(H) \), \( G \) has at most 3 edges between \( F' \) and \( H \). Since every such \( F' \) is either a component of \( G_1 - V(H) \), or obtained from a component \( F \) of \( G_1 - V(H) \) by adding some pendant vertices to \( F \), \( H \) is also a J-subgraph of \( G \). By Lemma 2.2, \( L(G) \) is hamiltonian, and so Conjecture 1 holds.

3. The existence of J-subgraphs and hamiltonian line graphs.

We need a few more terms. For a graph \( G \) with \( X \subseteq E(G) \), the contraction \( G/X \) is the graph obtained from \( G \) by identifying the two ends of each edge in \( X \) and then by deleting the resulting loops. We write \( G/e \) for \( G/[e] \) when \( X = \{e\} \). A graph \( G \) is collapsible if for every subset \( R \subseteq V(G) \) with \( |R| \) even, \( G \) has a spanning connected subgraph \( H_R \) such that \( O(H_R) = R \). Catlin introduced collapsible graphs in [3] mainly for finding graphs with spanning eulerian subgraphs. Catlin in [3] showed that every graphs \( G \) has a unique collection of maximal collapsible subgraphs \( H_1, H_2, \ldots, H_c \). The reduction of \( G \) is the graph \( G' \) obtained from \( G \) by contracting all nontrivial maximal collapsible subgraphs of \( G \). If a vertex \( v \) in the reduction \( G' \) is the contraction image of a nontrivial subgraph \( H \) of \( G \), then we say that \( v \) is a nontrivial vertex.

Theorem 3.1 (Catlin [3]). Let \( G \) be a connected graph. Each of the following holds:

(i) If \( G \) is collapsible, then \( G \) has a spanning eulerian subgraph.
(ii) If \( G' \) is the reduction of \( G \), then either \( G' = K_1 \) or \( \delta(G') \leq 3 \).

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(iii) Cycles of length at most 3 are collapsible.

Lemma 3.2. Let G be a connected graph, and let e = uv ∈ E(G) be a cut edge of G. If one component of G − e has a J-subgraph, then G has a J-subgraph.

Proof. Let G₁ be a component of G − e such that v ∈ V(G₁). By assumption, G₁ has a J-subgraph H. As v is adjacent to at most one component F of G₁ − V(H) in G, H is a J-subgraph of G also.

Corollary 3.3. Let G be a connected graph, and let v ∈ D₁(G). If G − v has a J-subgraph, then G has a J-subgraph.

Lemma 3.4. Let G be a connected graph and let v ∈ V(G) be a vertex of degree 2 that is incident with edges e, e' ∈ E(G). If G/e has a J-subgraph, then G has a J-subgraph.

Proof. Let G₂ = G/e. By assumption, G₂ has a J-subgraph H'. Let

\[ H = \begin{cases} G[E(H')] & \text{if } e' \notin E(H') \\ G[E(H') \cup \{e\}] & \text{if } e' \in E(H'). \end{cases} \]

Note that H is an eulerian subgraph of G and that v ∈ V(H) if and only if F is also a component of G₂ − V(H'), therefore there are at most 3 edges between F and H, and so H is a J-subgraph of G. If v ∈ V(H), then either e' ∈ E(G₂[V(H')]) and so {v} is a component of G − V(H), or e' ∈ E(F') for some component F' of G₂ − V(H') and so v is in the component F = G[E(F) \cup \{e\}] of G − V(H). In any case, since H' is a J-subgraph of G₂, H is a J-subgraph of G.

For a graph G, we define Ĝ to be the graph obtained from G by repeatedly deleting all vertices of degree 1 in G until none are left (the resulting graph at this stage will be called Ĝ₁), and then by eliminating all vertices of degree 2 by repeatedly contracting as edge that is incident with a vertex of degree 2 until none are left. Note that if Ĝ₁ is 2-edge-connected and is not a cycle, then Ĝ has minimum degree 3.

Proposition 3.5. If Ĝ has a J-subgraph, then G has a J-subgraph.

Proof. If follows by combining Corollary 3.3 and Lemma 3.4.

Proposition 3.6. For any graph G, if the reduction Ĝ' of Ĝ has a nontrivial vertex of degree at most 3 in Ĝ', then G has a J-subgraph.

Proof. By Proposition 3.5, it suffices to show that Ĝ has a J-subgraph. Thus we may assume that Ĝ = Ĝ'. Let u ∈ V(Ĝ') = V(G') be a nontrivial vertex that has degree at most 3 in
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Therefore we as assumed that $G$ is not a tree, and so by Corollary 3.7, $G$ has a J-subgraph $H$. By Lemma 2.2, $L(G)$ is hamiltonian. This proves Corollary 3.8.

Corollary 3.9. For any graph $G$, if $L^2(G)$ is 4-connected, then $L^2(G)$ is hamiltonian.

Proof. We shall show that $L(G)$ has a J-subgraph. We claim first that any vertex of degree at most 3 in $L(G)'$ (the reduction of $L(G)$) must be nontrivial. Let $v \in V(L(G)')$ be a vertex of degree at least 3. If $v$ has degree at most 2 and is trivial, then $v$ should have been deleted, or eliminated, in forming $L(G)'$, a contradiction. Hence we may assume that $v$ has degree 3 in $L(G)'$. If $v$ is trivial, then $v \in V(L(G))$ is a vertex of degree 3. Since $L(G)$ does not have an induced $K(1,2)$ (see [4], page 74), the two edges incident with $v$ must be in a 3-cycle $C$ (say) of $L(G)$. By (iii) of Theorem 3.1, $v$ must be lying in a maximal collapsible subgraph $H$ containing $C$ in $L(G)$, contrary to the assumption that $v$ is a trivial vertex in the reduction $L(G)'$. Hence we have proved the claim that any vertex of degree at most 3 in $L(G)'$ must be nontrivial. By Proposition 3.6, $L(G)$ has a J-subgraph. By Lemma 2.2 and by the assumption that $L^2(G)$ is 4-connected, $L^2(G)$ must be hamiltonian.

References


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\( \tilde{G} \): Let \( H_u' \) denote the preimage of \( u \) in \( G \). Note that by the definition of a reduction, \( H_u' \) is a collapsible subgraph of \( G \). By (i) of Theorem 3.1, \( H_u' \) has a nontrivial eulerian subgraph \( H_u \) spanning \( H_u' \). Since the degree of \( u \) is at most 3 in \( G' \), and since \( H_u \) spans \( H_u' \), for any component \( F \) of \( G - V(H_u) = G - V(H_u') \), there are at most 3 edges between \( F \) and \( H_u' \), and so \( H_u' \) is a J-subgraph of \( G \). This proves Proposition 3.6.

Corollary 3.7. Let \( G \) be a connected graph. If \( G \) is not a tree and if \( G \) does not have a pseudo-peripheral edge cut of size 3, then \( G \) has a J-subgraph.

Proof. By Lemma 3.2, we assume that \( \tilde{G}_1 \) is 2-edge-connected. If \( \tilde{G}_1 \) is a cycle, then this cycle is a J-subgraph of \( T \). Thus \( \tilde{G} \) has minimum degree at least 3.

Since \( G \) does not have any pseudo-peripheral edge cut of size 3, if \( \tilde{G} \) has a vertex \( u \) of degree 3, then in \( \tilde{G}_1 \), there must be a cycle \( C_u \) containing \( u \) such that all vertices in \( V(C_u - u) \) have degree 2 in \( G \). Such a cycle \( C_u \) is a J-subgraph of \( G \), and so we are done.

Therefore we assume that \( \delta(\tilde{G}) \geq 4 \). Let \( \tilde{G}' \) denote the reduction of \( \tilde{G} \). By (ii) of Theorem 3.1, either \( \tilde{G}' \) is collapsible (in that case \( \tilde{G}' = K_1 \)), or \( \delta(\tilde{G}') \geq 3 \). If \( \tilde{G}' \) is collapsible, then by (i) of Theorem 3.1, \( \tilde{G} \) has a spanning eulerian subgraph \( H \), which is a J-subgraph of \( G \). If \( \tilde{G} \) is not collapsible, then since \( \delta(\tilde{G}) \geq 4 \) and \( \delta(\tilde{G}') \leq 3 \), the preimage of a vertex \( u \) of degree at most 3 in \( \tilde{G}' \) must be a nontrivial collapsible subgraph \( H_u' \) of \( G \). By Proposition 3.6, \( G \) has a J-subgraph.

Corollary 3.8. If a connected graph \( G \) does not have vertices of degree 3, and if \( L(G) \) is 4-connected, then \( L(G) \) is hamiltonian.

Proof. If \( G \) has a pseudo-peripheral edge cut \( X \) of size 3, then the three edges in \( X \) must be incident with a vertex \( v \) such that the component of \( G - X \) containing \( v \) is a path \( P_v \) with \( v \) being an end of the path. If \( P_v \) has length zero, then \( v \) is a vertex of degree 3 in \( G \), contrary to the assumption that \( G \) does not have any vertices of degree 3. If \( P_v \) has length at least 1, then \( X \) would be a vertex cut of \( L(G) \), contrary to the assumption that \( L(G) \) is 4-connected. Therefore \( G \) does not have any pseudo-peripheral edge cut of size 3.

If \( G \) is a tree, then any edge of \( G \) is a cut edge of \( G \). Since \( L(G) \) is 4-connected, \( G \) does not have an edge \( e \) such that both components of \( G - e \) have an edge. Therefore, \( G \) is \( K(1,n-1) \) where \( n = |V(G)| \geq 4 \). In this case, \( L(G) \) is a complete graph and so is hamiltonian.