Small Circuit Double Covers of Cubic Multigraphs

HONG-JIAN LAI

Department of Mathematics, West Virginia University,
Morgantown, West Virginia 26506

XINGXING YU

School of Mathematics, Georgia Institute of Technology,
Atlanta, Georgia 30332

AND

CUN-QUAN ZHANG*

Department of Mathematics, West Virginia University,
Morgantown, West Virginia 26506

Received October 2, 1990

Let $G$ be a two-connected graph. A family $F$ of circuits of $G$ is called a circuit
double cover (CDC) if each edge of $G$ is contained in exactly two circuits of $F$. In
this paper, we show that if a simple cubic graph $G$ ($G \neq K_4$) of order $n$ has a CDC,
then $G$ has a CDC containing at most $n/2$ circuits. This result establishes the
equivalence of the circuit double cover conjecture (due to Szekeres, Seymour)
and the small circuit double cover conjecture (due to Bondy) for any cubic graph.
Actually, a stronger result is obtained in this paper for all loopless cubic graphs.
Another result in this paper establishes an upper bound on the size of any CDC of
a cubic graph. © 1994 Academic Press, Inc.

1. INTRODUCTION

We follow the terminology and notations of [BM]. Unless otherwise
stated, the graphs considered in this paper are connected and loopless
(parallel edges are allowed).

1.1. Circuit Double Covers

Let $G$ be a connected cubic graph of order $n$. If $G$ has a family $F$ of
circuits such that each edge of $G$ is contained in exactly two circuits of $F$,
then $F$ is called a circuit double cover or, for short, a CDC, of $G$.

* The research of this author was partially supported by National Science Foundation
under the Grant DMS-8906973.
The following conjectures are well known. The main result of this paper will establish their equivalence.

**Conjecture A** (Szekeres [SZ], Seymour [S], or see [J1, J2]). Every two-connected cubic graph has a circuit double cover.

**Conjecture B** (Bondy [B1]). Every two-connected simple cubic graph $G$ of order $n$ has a circuit double cover consisting of at most $n/2$ circuits if $G \neq K_4$.

In the following theorem, we establish an upper bound on the size of any CDC of a cubic graph.

**Theorem 1.** If $F$ is a circuit double cover of a connected cubic graph $G$ of order $n$, then $|F| \leq n/2 + 2$.

### 1.2. Small Circuit Double Covers

A loopless cubic graph with two vertices and three parallel edges is denoted by $K_2^{(3)}$ and a complete graph with four vertices is denoted by $K_4$.

A connected graph with four vertices, two of which are of degree one and two of which are of degree three, is called a $\phi$-graph (see Fig. 1). Let $G$ be a loopless cubic graph. A *blistering* of $G$ is constructed by recursively replacing edges by $\phi$-graphs (see Fig. 2). For the sake of convenience, we say that a graph $G$ is a blistering of itself (replacing edges by $\phi$-graphs zero times). Figure 2 illustrates this concept with some examples: a blistered $K_2^{(3)}$ and a blistered $K_4$. (Note that this definition of a blistered graph is different from the definition originally given in [AGZ]).

A CDC $F$ of a connected cubic graph $G$ is called a *small circuit double cover* or, for short, an SCDC, of $G$, provided that

1. $|F| \leq n/2 + 2$, if $G$ is a blistered $K_2^{(3)}$;
(ii) \(|F| \leq n/2 + 1\), if \(G\) is a blistered \(K_4\);

(iii) \(|F| \leq n/2\), otherwise.

By the definition of blistered graphs, \(G = K_4^{(3)}\) and \(G = K_4\) are included in (i) and (ii), respectively. Note that the definition of a small circuit double cover is an extension of the original definition of SCDC introduced by Bondy [B1]. Let \(\Gamma_3\) be the set of all two-connected cubic graphs, let \(\Gamma_{\text{CDC}}\) be the set of all connected cubic graphs admitting a CDC and \(\Gamma_{\text{SCDC}}\) be the set of all connected cubic graphs admitting an SCDC. Obviously,

\[
\Gamma_{\text{SCDC}} \subseteq \Gamma_{\text{CDC}} \subseteq \Gamma_3.
\]

The following problem is a refinement of Conjecture B.

**Conjecture B'.** Every two-connected cubic graph has a small circuit double cover (that is, \(\Gamma_{\text{SCDC}} = \Gamma_3\)).

**Previous Results** [LYZ]. (i) If every two-connected cubic graph has a circuit double cover, then every two-connected cubic graph has a small circuit double cover (that is, if \(\Gamma_{\text{CDC}} = \Gamma_3\) then \(\Gamma_{\text{SCDC}} = \Gamma_3\)).

(ii) Every two-connected cubic graph containing no subdivision of the Petersen graph has a small circuit double cover. (It was proved in [AZ] that every such graph has a circuit double cover.)

(iii) Every three-edge-colorable cubic graph has a small circuit double cover. (The case of hamiltonian cubic graphs was originally proved in [Y].)

Some related results about the small circuit double cover also can be found in [B1, B2, LH, SK], etc. The following problem was proposed in [LYZ] and is solved in this paper. One of the techniques that we use here is similar to one employed by Goddyn [G] in showing that the girth of a smallest counterexample to Conjecture A is at least seven.
Theorem 2. If a two-connected cubic graph $G$ has a circuit double cover, then $G$ has a small circuit double cover (that is, $\Gamma_{SCDC} = \Gamma_{CDC}$).

1.3. Strong Embedding of Cubic Graphs

A graph is said to be embedded in a surface $S$ (a closed two-manifold) if it can be drawn in $S$ so that edges intersect only at their common vertices. If $G$ is embedded in a surface $S$, then we regard $G$ as a topological subspace of $S$ and each component of $S \setminus G$ is called a face of the embedding. An embedding of $G$ in $S$ is a strong-embedding if every face is homeomorphic to the open disk and each face boundary is a circuit of $G$. (A strong embedding is also sometimes called a circular embedding, see [J1, J2].) As indicated by Jaeger [J1], when $G$ is a cubic graph, every circuit double cover $F$ is the system of face boundaries of a strong embedding in some surface $S$. The surface $S$ is said to be induced by the CDC $F$.

A recent result due to Richter, Seymour, and Širáň [RSS] asserts that every three-connected planar graph has a strong embedding in some non-spherical surface. For cubic graphs, the following corollary of Theorem 2 generalizes this result, assuming the truth of the CDC conjecture.

Corollary 3. Every two-connected cubic simple graph $G$ has a strong embedding in some non-spherical surface if and only if $G$ has a CDC.

Proof. Let $F$ be an SCDC of $G$ and let $S$ be the surface induced by $F$. Denote the Euler characteristic of $S$ by $k(S)$. Then by Euler's formula,

$$|V(G)| + |F| - |E(G)| = k(S).$$

Since $G$ is cubic $|E(G)| = 3|V(G)|/2$ and by Theorem 2, $|F| \leq |V(G)|/2 + 1$, unless $G$ is a blistered $K_2^{(3)}$. It follows that $k(S) \leq 1$ if $G$ is simple. The surface $S$ must thus be non-spherical.

1.4. Small circuit 2k-Covers of Cubic Graphs

A two-edge-connected graph $G$ is said to be circuit $2k$-coverable if $G$ has a family $F$ of circuits such that each edge of $G$ is contained in precisely $2k$ circuits of $F$. This family of circuits is called a circuit $2k$-cover of $G$; when $k = 1$, we have a circuit double cover. Unlike the circuit double cover conjecture, which is still open, all other circuit $2k$-cover problems (for $k \geq 2$) have been solved. The circuit four-cover theorem is due to Bermond, Jackson, and Jaeger (see [BJJ]) and the circuit six-cover theorem is due to Fan (see [F]). As mentioned in [F], the existence of a circuit $2k$-cover (for $k \geq 2$) of any two-edge-connected graph is immediately implied by the above two results. The small circuit double cover conjecture for cubic graphs is verified in Theorem 2, assuming the existence of a circuit double
cover. The result below generalizes Theorem 2 to $2k$-coverings. Because of the theorems of Bermond, Jackson, and Jaeger and Fan, the assumption of the existence of a $2k$-cover for a graph can be dropped. By imitating the proof of Theorem 2 and by dropping the assumption that $G$ has a circuit double cover, we obtain the following theorem.

**Theorem 4.** Let $G$ be a two-edge-connected cubic graph with $n$ vertices, let $k \geq 2$ be an integer, and let $SC_k(G)$ denote the number of circuits in a smallest circuit $2k$-cover of $G$. Then

1. $SC_k(G) \leq k(n/2 + 2)$, if $G$ is a blistered $K_2^{(3)}$;
2. $SC_k(G) \leq k(n/2 + 1)$, if $G$ is a blistered $K_4$;
3. $SC_k(G) \leq k(n/2)$, for all other graphs.

2. Circuit Double Covers of Cubic Graphs

For any connected cubic graph $G$ admitting a CDC, Theorem 2 establishes an upper bound on the size of a smallest CDC of $G$ (a max–min problem), while the following theorem provides an upper bound for all CDCs of $G$.

**Theorem 1.** If $F$ is a CDC of a connected cubic graph $G$ of order $n$, then $|F| \leq n/2 + 2$.

**Proof.** It is well known that the circuit space of a connected graph with $n$ vertices and $m$ edges has dimension $m - n + 1$. The addition operation in this vector space is the symmetric difference (binary sum) of edge sets of the circuits. The CDC $F$ is a subset of the circuit space of $G$. Hence the rank $r(F)$ of $F$ (the maximum number of independent circuits in $F$) satisfies the inequality

$$r(F) \leq \frac{3n}{2} - n + 1 = \frac{n}{2} + 1.$$

Now we claim that $r(F) = |F| - 1$. For otherwise, there is a proper subset $F'$ of $F$ such that the binary sum $\sum_{C \in F'} E(C) = \emptyset$. The circuits of $F'$ induce a proper subgraph $H$ of $G$, and each edge of $H$ is covered twice by $F'$. Let $e$ be any edge of $G \setminus E(H)$ with at least one end in $H$. Since $G$ is cubic, any circuit in $G$ containing $e$ must use at least one edge in $H$. This is a contradiction since $e$ must be covered twice by the CDC $F$. Hence $|F| - 1 \leq n/2 + 1$, and so $|F| \leq n/2 + 2$. □
An Alternative Proof (L. Goddyn and B. Richter, personal communication). Each circuit of $F$ can be considered as the boundary of a disk. The graph $G$ is therefore embedded in a surface $S$ established by joining all these disks at the edges of $G$. Since the Euler characteristic of $S$ is not greater than two, by Euler’s formula, we have that

$$|\bar{V}(G)| + |F| - |E(G)| \leq 2.$$ 

Note that $|V(G)| = n$ and $|E(G)| = 3n/2$ since $G$ is cubic. Therefore, no circuit double cover $F$ of $G$ contains more than $n/2 + 2$ circuits. 

Actually, the alternative proof gives a generalization of Theorem 1.

**Theorem 1'.** If $F$ is a CDC of a connected cubic graph $G$ of order $n$, then $|F| \leq n/2 + k(S)$, where $S$ is the surface induced by $F$ and $k(S)$ is the Euler characteristic of the surface $S$.

3. Small Circuit Double Covers of Cubic Graphs

If $G$ is a loopless graph in which the degree of each vertex is either two or three, then the cubic graph that is homeomorphic to $G$ is called the background graph of $G$ and is denoted by $B(G)$ (see Fig. 3). A trivial cut $X$ of a graph $G$ is an edge-cut of $G$ such that one component of $G \setminus X$ is a single vertex.

**Theorem 2.** If a two-connected cubic graph $G$ has a circuit double cover, then $G$ has a small circuit double cover (that is, $\Gamma_{SCDC} = \Gamma_{CDC}$).

**Proof.** Assume that $\Gamma_{SCDC} \neq \Gamma_{CDC}$. Let $G$ be a smallest graph in $\Gamma_{CDC} \setminus \Gamma_{SCDC}$. Since $K_2^{(3)}$ and $K_4$ belong to $\Gamma_{SCDC}$, $G \neq K_2^{(3)}$, $K_4$. Let $|V(G)| = n$. 

![Figure 3](image-url)
I. G has no two-cut. Assume that G has a two-cut. Choose a two-cut $X = \{xx', yy'\}$, where $G_1$ and $G_2$ are two components of $G \setminus X$ and $x, y \in V(G_1)$, $x', y' \in V(G_2)$ such that $G_1$ is as small as possible. Note that $x \neq y$, $x' \neq y'$ since $G$ is cubic and has no cut-edge (see Fig. I-1). Let $H_1 = G_1 \cup \{e\}$ and $H_2 = G_2 \cup \{e'\}$, where $e$ and $e'$ are new edges joining $x$ and $y$, $x'$ and $y'$, respectively (see Fig. I-2). If $G$ is not simple then, by the choice of $X$, $|V(G_1)| = 2$ and $H_1 = K_2^{(3)}$. Let $F$ be a CDC of $G$. Let $C_1$ and $C_2$ be the two circuits of $F$ containing $xx'$ and $yy'$. Let $C_i'$ be the segment of $C_i$ of $G_i$ between $x$ and $y$, together with the edge $e$, $i = 1, 2$. Then

$$\{C \in F : C \neq C_1, C_2 \text{ and } E(C) \cap E(H_1) \neq \emptyset \} \cup \{C_1', C_2'\}$$

is a CDC of $H_1$. That is, $H_1 \in \Gamma_{CDC}$. By the inductive hypothesis, $H_1 \in \Gamma_{SCDC}$. Similarly, $H_2 \in \Gamma_{SCDC}$.

Let $F_1^*$ and $F_2^*$ be SCDCs of $H_1$ and $H_2$, respectively. Let $D_1', D_2''$ (respectively, $D_2', D_2''$) be the circuits of $F_1^*$ (respectively, $F_2^*$) containing the new edge $e = xy$ (respectively, $e' = x'y'$).

Let $D' = D_1' \Delta D_2'' \Delta C_4$ and $D'' = D_1'' \Delta D_2' \Delta C_4$, where $C_4$ is a circuit $xx'y'yx$ and $\Delta$ is the symmetric difference. Then

$$F^{**} = [F_1^* \cup F_2^* \cup \{D', D''\}] \setminus \{D_1', D_1'', D_2', D_2''\}$$

\begin{center}
\includegraphics[width=0.4\textwidth]{fig1-2.png}
\end{center}

\textbf{Figure I-2}
is a CDC of $G$ consisting of $\vert F_1^* \vert + \vert F_2^* \vert - 2$ circuits. Note that $\vert V(G) \vert = \vert V(H_1) \vert + \vert V(H_2) \vert$, so $F^{**}$ is an SCDC of $G$ if $\vert F_1^* \vert + \vert F_2^* \vert \leq \eta/2 + 2$. Thus, by the assumption, we have that $\vert F_1^* \vert + \vert F_2^* \vert \geq \eta/2 + 3$. That is, one of $H_1$, $H_2$ must be a blistered $K_2^{(3)}$ and the other must be either a blistered $K_2^{(3)}$ or a blistered $K_4$. It is evident that any blistered $K_2^{(3)}$ has at least two distinct pairs of parallel edges. If $H_i$ (for $i = 1$ or $2$) is a blistered $K_2^{(3)}$, then one pair of parallel edges of $H_i$ must originally exist in $G$ and therefore $G$ is not simple. By the choice of the edge-cut $X$ and the component $G_1$, we can see that $\vert V(G_1) \vert = 2$ and $H_1 = K_2^{(3)}$. Thus $G$ is a blistered graph of $H_2$. Furthermore, $G$ is a blistered $K_2^{(3)}$ (respectively, a blistered $K_4$) if $H_2$ is a blistered $K_2^{(3)}$ (respectively, a blistered $K_4$). Note that blistering one edge adds two vertices and requires exactly one more circuit to double cover the new edges. Therefore the CDC $F^{**}$ constructed above is an SCDC of $G$. This is a contradiction. Thus $G$ has no two-circuit and is simple.

II. $G$ has no non-trivial three-cut. Suppose that $G$ has a non-trivial three-cut $X = \{xx', yy', zz'\}$ with two non-trivial components $G_1$ and $G_2$ (see Fig. II-1). Since $G$ is cubic and has no two-cut, $X$ is a matching of $G$. Let $H_1$ (respectively, $H_2$) be the graph constructed from $G$ by contracting all edges in $G_2$ (respectively, $G_1$), and denote the new vertex in $H_1$ (respectively, $H_2$) by $w_2$ (respectively, $w_1$) (see Fig. II-2). Let $F$ be a CDC of $G$. Let $C_{xy}$ be the circuit of $F$ containing the edges $xx'$ and $yy'$; the circuits $C_{xz}$ and $C_{yz}$ of $F$ are defined similarly. Let $C'_{xy}$ (respectively, $C'_{xz}$ and $C'_{yz}$) be the circuit constructed from $C_{xy}$ (respectively, $C_{xz}$ and $C_{yz}$) by contracting all edges in $G_2$. Then

$$\{C: C \in F \text{ and } E(C) \subseteq E(G_1)\} \cup \{C'_{xy}, C'_{xz}, C'_{yz}\}$$

is a CDC of $H_1$. By the inductive hypothesis, $H_1 \in \mathcal{F}_{\text{SCDC}}$. Similarly, $H_2 \in \mathcal{F}_{\text{SCDC}}$. Let $F_1^*$ and $F_2^*$ be SCDCs of $H_1$ and $H_2$, respectively. Let $D'_{xy}$ be the circuit of $F_1^*$ containing the edges $xw_2$ and $yw_2$; define $D'_{xz}, D'_{yz}$ similarly. Let $D''_{xy}$ be the circuit of $F_2^*$ containing the edges $x\prime w_1$ and $y\prime w_1$; define $D''_{xz}, D''_{yz}$ similarly. Let

$$D_{xy} = [D'_{xy} \cup D''_{xy} \cup \{xx', yy'\}] \setminus \{xw_2, yw_2, x\prime w_1, y\prime w_1\};$$

define $D_{xz}, D_{yz}$ similarly. Then

$$F^{**} = [F_1^* \cup F_2^* \cup \{D_{xy}, D_{xz}, D_{yz}\}] \setminus \{D'_{xy}, D'_{xz}, D'_{yz}, D''_{xy}, D''_{xz}, D''_{yz}\}$$

is a CDC of $G$. Note that $G$ is simple and $X$ is a three-matching. Thus both $H_1$ and $H_2$ are simple and neither $H_1$ nor $H_2$ is a blistered $K_2^{(3)}$. Therefore, by the inductive hypothesis,

$$|F_1^*| \leq \frac{|V(H_1)|}{2} + 1, \quad |F_2^*| \leq \frac{|V(H_2)|}{2} + 1.$$
Since $|F^{**}| = |F_1^*| + |F_2^*| - 3$ and $|V(G)| = |V(H_1)| + |V(H_2)| - 2$, $F^{**}$ is an SCDC of $G$, a contradiction.

Hence we can see that $G$ is triangle-free.

III. No CDC of $G$ contains a four-circuit. Let $F$ be a CDC of $G$. Assume that $F$ has some circuit of length four, say $C = uvwx$. Let $u'u, v'v, w'w,$ and $x'x$ be four edges of $E(G) \setminus E(C)$. Note that since $G$ has no three-circuit, either $\{u'u, v'v, w'w, x'x\}$ is a four-matching or (without loss of generality) $u' = w'$. If $u' = w'$, then $G$ has a three-cut consisting of $vv'$, $xx'$ and the edge incident with $u'$ other than $uu'$ and $ww'$. Since $G$ has no non-trivial three-cut, we have that $v' = x'$ and therefore $G = K_{3,3}$ for which the theorem holds. So we assume that $\{u'u, v'v, w'w, x'x\}$ is a four-matching of $G$. Let $C_1$ be the circuit of $F$ containing $u'w'$, $C_2$ be the circuit of $F$ containing $v'vw'$, $C_3$ be the circuit of $F$ containing $w'wxx'$, and $C_4$ be the circuit of $F$ containing $x'xuu'$ (see Fig. III-1).

Case 1. $C_2 \neq C_4$ (or, symmetrically, $C_1 \neq C_3$). Let $D = C_4AC$. Then $[F \setminus \{C_4, C\} \cup \{D\}]$ is a CDC of $H = G \setminus \{ux\}$ (see Fig. III-2). Since the background graph $B(H) \in \Gamma_{SCDC}$, by the inductive hypothesis, $B(H) \in \Gamma_{SCDC}$. Let $F^*$ be an SCDC of $B(H)$. Since $G$ is triangle-free, $B(H)$ is simple and not a blistered graph. Furthermore, $B(H)$ is neither $K_{2,3}$ nor $K_4$ because $B(H)$ contains at least six vertices $\{u', v, v', w, w', x'\}$. Thus $F^*$ consists of at most $|V(B(H))|/2 = (n - 2)/2$ circuits.
Subcase 1. $F^*$ has a circuit $D_1$ containing the path $u'uvwxv'$ (see Fig. III-3). Let $D_2 = D_1 \Delta C$. Then $[F^* \setminus \{D_1\}] \cup \{D_2, C\}$ is an SCDC of $G$, a contradiction.

Subcase 2. The path $u'uvwxv'$ does not belong to any circuit of $F^*$. Then the circuits of $F^*$ containing $v$ or $w$ must be of the following four types $E_1$, $E_2$, $E_3$, $E_4$ (see Fig. III-4): $E_1$ contains $u'uvwxv'$, $E_2$ contains $u'uvv'$, $E_3$ contains $x'xvwv'$, and $E_4$ contains $x'xww'$.  

(i) If $E_2 \neq E_4$, let $E'_2 = E_2 \Delta C$, $E'_3 = E_3 \Delta C$ (note that $E_2 \neq E_3$ because $E_2 \cup E_3$ has a vertex of degree three). Then (see Fig. III-5) $[F^* \setminus \{E_2, E_3\}] \cup \{E'_2, E'_3\}$ is an SCDC of $G$. 

![Figure III-2](image-url)
(ii) If $E_2 = E_4$, then the union of the circuit $E_2$ and its chord $ux$ can be covered by two circuits $E_5$ and $E_6$ such that $E_5 \Delta E_6 = E_2$ and $E_5 \cap E_6 = \{ux\}$. Thus $[F^* \setminus \{E_2\}] \cup \{E_5, E_6\}$ is an SCDC of $G$, a contradiction.

Case 2. $C_2 = C_4$ and $C_1 = C_3$ (refer to Fig. III-1). We claim that either $u'v', w'x' \notin E(G)$ or $v'w', x'u' \notin E(G)$. Without loss of generality, assume to the contrary that $v'w', w'x' \in E(G)$. Then the edges $\{uu', v'v'', x'x''\}$, where $v'' \in N(v') \setminus \{v, w'\}$, $x'' \in N(x') \setminus \{x, w'\}$, form a three-edge-cut of $G$ (see Fig. III-6). Thus $G$ is a three-cube since $G$ has no non-trivial three-cut. It is very easy to check that the theorem holds for the three-cube.

Suppose that $u'v', w'x' \notin E(G)$. Then the background graph of $H' = G \setminus \{ux, vw\}$ is simple. Let $D = C_4 \Delta C$, a union of circuits. Then
\[ \{F \backslash \{C_4, C\}\} \cup \{D\} \] is a CDC of \( H' = G \backslash \{ux, vw\} \). By the inductive hypothesis, the background graph \( B(H') \in \Gamma_{\text{SCDC}} \). Let \( F^{**} \) be an SCDC of \( B(H') \). Note that \( B(H') \) contains at least four vertices \( \{u', v', w', x'\} \). If \( B(H') = K_4 \), then the graph \( G \) is illustrated in Fig. III-7; an SCDC can easily be found in this graph. Since \( B(H) \) is simple, we may thus assume that \( B(H') \) is neither a blistered \( K_2^{(3)} \) nor a blistered \( K_4 \). Hence \( |F^{**}| \leq |V(B(H'))|/2 = n/2 - 2 \). Let \( D_1, D_2 \) be the circuits of \( F^{**} \) containing the path \( u'uuv' \) and \( D_3, D_4 \) be circuits of \( F^{**} \) containing the path \( w'wxx' \) (see Fig. III-8).
Subcase 1. \( \{D_1, D_2\} \cap \{D_3, D_4\} = \emptyset \). Let \( D'_1 = D_1 \Delta C \) and \( D'_1 = D_3 \Delta C \) (see Fig. III-9). Then \([F^{**}\{D_1, D_3\}] \cup \{D'_1, D'_3\}\) is an SCDC of \( G \), a contradiction.

Subcase 2. \( \{D_1, D_2\} \cap \{D_3, D_4\} \neq \emptyset \). Without loss of generality, suppose that \( D_1 = D_3 \). The symmetric difference of \( D_1 \) and \( C \) is the union of at most two circuits since \( D_1 \backslash C \) has only two segments. Thus \([F^{**}\{D_1\}] \cup \{D_1 \Delta C, C\}\) (see Fig. III-10) is an SCDC of \( G \) consisting of at most \( n/2 \) circuits.

IV. No CDC of \( G \) contains a five-circuit. Assume that the CDC \( F \) of \( G \) contains a circuit \( C \) of length five. Let \( C = x_1 x_2 \cdots x_5 x_1 \) and \( y_i \) be the neighbor of \( x_i \) other than \( x_{i-1} \) and \( x_{i+1} \) (mod 5). Since \( G \) is triangle-free,
$\{x_1, \ldots, x_5\}$ and $\{y_1, \ldots, y_5\}$ are disjoint. Denote the circuit of $F$ containing the path $y_i x_i x_{i+1} y_{i+1}$ (mod 5) by $C_i$ (see Fig. IV-1).

(i) Since $F$ is a CDC of $G$, $C_i \neq C_{i+1}$ for $i = 1, \ldots, 5$ (mod 5). Hence $\{C_1, \ldots, C_5\}$ is a set of at least three distinct circuits, and one element of it must be distinct from all others.

(ii) By (i), we assume, without loss of generality, that $C_5 \neq C_1, C_2, C_3,$ and $C_4$. Let $D = C_5 \Delta C$ (see Fig. IV-2). Then $[F \backslash \{C_5, C\}] \cup \{D\}$ is a CDC of $H = G \backslash \{x_1 x_5\}$. By the inductive hypothesis, $B(H) \in \Gamma_{\text{SCDC}}$. Let $F^*$ be an SCDC of $B(H)$.

(iii) We claim that $|F^*| \leq |V(B(H))|/2$. By the inductive hypothesis, we only need to show that $B(H)$ is a simple graph other than $K_2^{(3)}$ and $K_4$. Since $G$ is triangle-free, $B(H)$ must be simple and $|\{y_1, \ldots, y_5\}| \geq 3$. Thus $B(H)$ has at least six distinct vertices $(x_2, x_3, x_4, y_1, \ldots, y_5)$. This excludes the possibility that $B(H) = K_4$, so our claim holds.

(iv) Since $\{x_i y_i: 1 \leq i \leq 5\}$ is an edge-cut, every circuit in $F^*$ contains an even number of edges in $\{x_i y_i: 1 \leq i \leq 5\}$ and so the edge set $\{x_i y_i: 1 \leq i \leq 5\}$ of $H$ is covered by at most five distinct circuits of $F^*$. Let $D_1$ and $D_2$ be the circuits of $F^*$ containing the path $y_2 x_5 x_4$ and let $E_1$ and $E_2$ be the circuits of $F^*$ containing $y_1 x_1 x_2$.

We claim that $D_1, D_2, E_1, E_2$ are distinct. It is trivial that $D_1 \neq D_2$ and $E_1 \neq E_2$. Assume that $D_1 = E_1$. Then the union of the circuit $D_1$ and its chord $x_5 x_1$ can be covered by two circuits $D'$ and $D''$ such that $D' \cap D'' = \{x_5 x_1\}$ and $D' \Delta D'' = D_1$. Thus $[F^* \backslash \{D_1\}] \cup \{D', D''\}$ is an SCDC of $G$. This contradicts the assumption that $G$ is a counterexample to the theorem.
(v) For $i = 1, 2$, let $D_i$ contain the path $y_5x_5x_4 \cdots x_{d_i}y_{d_i}$ for some $d_i \in \{2, 3, 4\}$ (note $d_i \neq 1$ by (iv)) and let $E_i$ contain the path $y_1x_1x_2 \cdots x_ey_e$ for some $e_i \in \{2, 3, 4\}$. It can be seen that $d_1 \neq d_2$, for otherwise the edges $x_{d_1}y_{d_1}, x_{d_1+1}x_{d_1}$ are covered twice by the circuits $D_1, D_2$ and the edge $x_{d_1}x_{d_1-1}$ cannot be covered by any circuit of $F^*$. Similarly, $e_1 \neq e_2$. Since $d_1, d_2, e_1, e_2 \in \{2, 3, 4\}$, we assume, without loss of generality, that $d_1 = e_1$.

(vi) Case 1. $d_1 = e_1 = 3$. The coverage of all edges incident with $x_1, ..., x_5$ by $F^*$ in $H$ can be easily determined and is illustrated in Fig. IV-3. The circuit $D$ in Fig. IV-3 contains the path $y_2x_2x_3x_4y_4$. Obviously $D$ is distinct from each of $D_1, D_2, E_1$, and $E_2$ since it intersects all of them. Let $D'_1 = D_1 \Delta C$ and $E'_1 = E_1 \Delta C$. Then $[F^* \setminus \{D_1, E_1\}] \cup \{D'_1, E'_1\}$ is an SCDC of $G$, a contradiction.

(vii) Case 2. $d_1 = e_1 = 2$ (or, symmetrically, $d_1 = e_1 = 4$). The coverage of all edges incident with $x_1, ..., x_5$ by $F^*$ in $H$ is illustrated in Fig. IV-4. The circuit $D$ in Fig. IV-4 contains the path $y_2x_2x_3x_4y_4$. Obviously the circuit $D$ is distinct from each of $D_1, D_2,$ and $E_2$, while it is possible that $D = E_1$. As in Case 1, let $D'_1 = D_1 \Delta C$ and $E'_1 = E_1 \Delta C$. If $D \neq E_1$, then $[F^* \setminus \{D_1, E_1\}] \cup \{D'_1, E'_1\}$ is an SCDC of $G$, a contradiction. Assume that $D = E_1$. Then $E'_1 = E_1 \Delta C = [E_1 \setminus \{x_1x_2, x_3x_4\}] \cup \{x_2x_3, x_4x_5, x_5x_1\}$. Thus $F^{**} = [F^* \setminus \{D_1, E_1\}] \cup \{D'_1, E'_1\}$ is a CDC of $H' = G \setminus \{x_3x_4\}$ (see Fig. IV-5). Here $E'_1 = E_1 \Delta C$ is the union of at most two circuits. And $|F^{**}| = |F^*| \leq (n-2)/2$ if $E'_1$ is a single circuit, or $|F^{**}| = |F^*| + 1 \leq n/2$ if $E'_1$ is the union of two disjoint circuits.
(a) If $E'_1$ is the union of two disjoint circuits $E^*$ and $E^{**}$, where $E^*$ contains the path $y_1, x_1, x_4, x_4, y_4$ and $E^{**}$ contains the path $y_2, x_2, x_3, y_3$, let $D^* = E^* \Delta C$ and $D^{**} = E^{**} \Delta C$. Then $[F^{**} \setminus \{E^*, E_2\}] \cup \{D^*, D^{**}\}$ is an SCDC of $G$.

(b) If $E'_1$ is a single circuit, then the union of the circuit $E'_1$ and its chord $x_3, x_4$ can be covered by two circuits $D^o$ and $D^{oo}$ such that $D^o \cap D^{oo} = \{x_3, x_4\}$ and $D^o \Delta D^{oo} = E'_1$, and $[F^{**} \setminus \{E'_1\}] \cup \{D^o, D^{oo}\}$ is an SCDC of $G$. Both contradict the assumption that $G$ has no SCDC.
V. Note that the number of edges of the cubic graph \( G \) is \( 3n/2 \), so the total length of all circuits of \( F \) is \( 3n \). That the length of each circuit of \( F \) is at least six implies that \( |F| \leq n/2 \). This is a contradiction and completes the proof of this theorem.

Acknowledgments

We thank R. Bruce Richter for his comment that led us to Corollary 3 and the referee for his helpful suggestions and corrections.

References


[LH] H. Li, Perfect path double covers in every simple graph, *J. Graph Theory* 14, No. 6 (1990), 645–650.


Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium