Graphs of diameter at most two

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Abstract
A graph $H$ is collapsible if for every even subset $W \subseteq V(H)$, $H$ has a spanning connected subgraph whose set of odd-degree vertices is $W$. In a graph $G$, there is a unique collection of maximal collapsible subgraphs, and when all of them are contracted, the resulting contraction of $G$ is a reduced graph. Reduced graphs have been shown to be useful in the study of super eulerian graphs, Hamiltonian line graphs, and double cycle covers, (see [2], [3], [4], [6]), among others. It has been noted that subdividing an edge of a collapsible graph may result in a noncollapsible graph. In this note we characterize the reduced graphs of elementary subdivision of collapsible graphs of diameter at most two. We also obtain a converse of a result of Catlin [3] when restricted to graphs of diameter at most two. The main result is used to study some Hamiltonian property of line graphs.

INTRODUCTION

We shall use the notation of Bondy and Murty [1], except for contractions, and we allow graphs to have multiple edges but loops are forbidden. We shall use $d_G(u,v)$ to denote the distance between the two vertices $u,v$ in $G$. When no confusion arises, we use $d(u,v)$ for $d_G(u,v)$. The diameter of $G$, denoted by $\text{diam}(G)$, is defined thus:

$$\text{diam}(G) = \max_{u,v \in V(G)} d(u,v).$$

The degree of a vertex $v$ in a graph $G$ will be denoted by $\text{deg}_G(v)$, or $\text{deg}(v)$. For integer $i \geq 0$, we define

$$D_i(G) = \{ v \in V(G) \mid \text{deg}_G(v) = i \}.$$
As in [1], \( \delta(G) \) denotes the minimum degree of \( G \), and \( \kappa(G) \) and \( \kappa'(G) \) denote the connectivity and the edge-connectivity of \( G \), respectively.

For a set \( X \subseteq E(G) \), we define the contraction \( G/X \) to be the graph obtained from \( G \) by contracting the edges of \( X \) and deleting all resulting loops. When \( H \) is a connected subgraph of \( G \), we use \( G/H \) for \( G/E(H) \).

In [2], Catlin defines the collapsible subgraphs. A graph \( G \) is collapsible if for every subset \( R \subseteq V(G) \) with \( |R| \) even, \( G \) has a subgraph \( \Gamma \) such that \( G - E(\Gamma) \) is connected and such that the set of odd-degree vertices of \( \Gamma \) is \( R \). The subgraph \( \Gamma \) is called an \( R \)-subgraph of \( G \). It is routine to show that \( G \) is collapsible if and only if for every subset \( W \subseteq V(G) \) with \( |W| \) even, \( G \) has a connected spanning subgraph whose set of odd-degree vertices is \( W \). In [2], Catlin showed that every vertex of a graph \( G \) is in a unique maximal collapsible subgraph of \( G \). The reduction of \( G \) is the graph obtained from \( G \) by contracting all nontrivial collapsible subgraphs of \( G \). A graph is called reduced if it is the reduction of some graph.

A graph \( G \) is eulerian if \( G \) is connected and every vertex of \( G \) has even degree. Note that the trivial graph \( K_1 \) is regarded as both collapsible and having spanning eulerian subgraphs.

**Theorem A** (Catlin [2]) Let \( G \) be a graph.

(a) \( G \) is reduced if and only if \( G \) has no nontrivial collapsible subgraphs.

(b) If \( G \) is reduced, then \( G \) is simple with \( \delta(G) \leq 3 \), and \( G \) contains no subgraph isomorphic to \( K_3 \).

(c) If each edge of a spanning tree of \( G \) is in a collapsible subgraph, then \( G \) is collapsible.

(d) If \( H \) is a connected subgraph of \( G \) and if \( G \) is collapsible, then \( G/H \) is collapsible; if \( H \) is a collapsible subgraph of \( G \) and if \( G/H \) is collapsible, then \( G \) is collapsible.

(e) \( G \) has a spanning eulerian subgraph if and only if the reduction of \( G \) has a spanning eulerian subgraph.

(f) If \( G \) is collapsible, then \( G \) has a spanning eulerian subgraph. \( \Box \).

Reduced graphs of diameter two are characterised in [6]. Let \( m, l \) be two positive integers. Let \( H_1 \cong K_{2,m} \) and \( H_2 \cong K_{2,l} \) be two complete bipartite graphs. Let \( v_1, u_1 \) be two nonadjacent vertices of degree \( m \) in \( H_1 \), and \( v_2, u_2 \) be two nonadjacent vertices of degree \( l \) in \( H_2 \). Let \( S_{l,m} \) denote the graph obtained from \( H_1 \) and \( H_2 \) by identifying \( v_1 \) and \( v_2 \), and by connecting \( u_1 \) and \( u_2 \) with a new edge \( u_1u_2 \). Note that \( S_{1,1} \) is the same as \( C_5 \), the 5-cycle.

**Theorem B** (Lai [7]) Let \( G \) be a reduced graph. If \( \text{diam}(G) = 2 \), then exactly one of the following holds:

(a) \( G \cong K_{1,t}, t \geq 2; \)
(b) $G \cong K_{2,t}, t \geq 2$;
(c) $G \cong S_{l,m}, l, m \geq 1$;
(d) $G$ is the Petersen graph. □

In [3], Catlin developed the idea of collapsible graphs. Let $H$ be a graph and let $\pi$ be a partition of $V(H)$ into two nonempty sets $V_1, V_2$. We shall denote this by $\pi = \langle V_1, V_2 \rangle$. Then $H$ is called $\pi$-collapsible if for every subset $R \subseteq V(H)$ of even cardinality, the following hold:

(i) if $|R \cap V_1|$ is odd, then $H$ has an $R$-subgraph;
(ii) if $|R \cap V_2|$ is even, then $H + e$ has an $R$-subgraph, for any newly added edge $e = v_1v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$.

As examples, the 2-cycle is collapsible, complete graphs of order at least 3 are collapsible; collapsible graphs are $\pi$-collapsible, for any partition $\pi$. However, the following example, as noted by Catlin in [3], is $\pi$-collapsible but not collapsible.

Example 1 Let $C_4 = v_1v_2v_3v_4v_1$ denote the 4-cycle, let $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_4\}$, and let $\pi = \langle V_1, V_2 \rangle$. Then $C_4$ is $\pi$-collapsible.

Suppose that $H$ is a $\pi$-collapsible subgraph of $G$ with $\pi = \langle V_1, V_2 \rangle$. Denote by $G/\pi$ the graph obtained from $G$ by identifying all vertices of $V_1$ to form a single vertex $v_1$, by identifying all vertices of $V_2$ to form a single vertex $v_2$, and by joining $v_1$ and $v_2$ with exactly one edge. This new edge is denoted by $e_\pi$.

Theorem C (Catlin [3]) Let $H$ be a $\pi$-collapsible subgraph of $G$. If $G/\pi$ is collapsible, then $G$ is collapsible. □

It is easy to construct examples to show the converse of Theorem C is not true in general:

Example 2 Let $H = K_{2,t}$ with $t \geq 2$. Let $e = xy \in E(H)$. Let $K$ be the graph with $V(K) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and $E(K) = \{x_1x_2, x_2x_3, x_3y_1, y_1y_2, y_2y_3, y_3x_1, x_2y_2, x_3y_3\}$.

Define $G(t)$ to be the graph obtained from $H - e$ and $K$ by identifying $x_1$ with $x$ and indentifying $y_1$ with $y$. It is shown in [3] that $K$ is collapsible. Note that every edge of $G(t)/K$ is in a 3-cycle and so by (c) and (d) of Theorem A, $G(t)$ is collapsible. Both $K$ and $G(t)$ contain a 4-cycle $x_2x_3y_3y_2x_2$. Let $\pi$ denote the bipartition of this 4-cycle. Then neither $K/\pi$ nor $G(t)/\pi$ is collapsible, for their reductions are $K_2$ and $K_{2,t}$, respectively.
MAIN RESULTS

Theorem 1 Let $G$ be a graph of diameter at most 2 and let $C_4$, the 4-cycle, be a nonspanning subgraph of $G$. Let $\pi$ denote the bipartition of $C_4$. Then $G$ is collapsible if and only if one of the following holds:

(a) $G/\pi$ is collapsible.
(b) $G$ is spanned by a subgraph $H \cong K_{2,n-2}$ such that there are two vertices in $D_2(H)$ adjacent in $G$.

Let $P_4 = x_1x_2x_3x_4$ denote a path of length 3. Let $H$ be a graph disjoint from $P_4$ that is isomorphic to a $K_{2,t}$ with $t \geq 2$ (respectively, an $S_{l,m}$ with $n \geq 2$ and $m \geq 1$). Let $xy \in E(H)$ be an edge of $H$ that is lying in a 4-cycle of $H$. Define $K_{2,t}^+$ (respectively, $S_{l,m}^+$) to be the graph obtained from $H$ and $P_4$ by identifying $x$ with $x_1$ and $y$ with $x_4$.

We say that an edge $e \in E(G)$ is subdivided when it is replaced by a path of length 2 whose internal vertex, denoted by $u(e)$, has degree 2 in the resulting graph, denoted by $G(e)$.

Theorem 2 Let $G$ be a collapsible graph of diameter at most 2, and let $e$ be an edge of $E(G)$. If $[G(e)]'$ denotes the reduction of $G(e)$, then exactly one of the following holds:

(a) $[G(e)]' \cong K_1$;
(b) $[G(e)]' \cong K_{2,t}$, $t \geq 2$;
(c) $[G(e)]' \cong S_{l,m}$, $l \geq 2$, $m \geq 1$;
(d) $[G(e)]' \cong K_{2,t}^+$, $t \geq 2$;
(e) $[G(e)]' \cong S_{l,m}^+$, $l \geq 2$, $m \geq 1$.

THE PROOF OF THEOREM 1

Lemma 1 Let $n \geq 5$ be an integer and let $H$ be a graph isomorphic to a $K_{2,n-2}$. Let $G$ be a spanning supergraph of $H$ such that there is an edge incident with two vertices in $D_2(H)$, then for any edge $e \in E(G)$, $G(e)$ is collapsible.

Proof: Let $x, y \in D_2(H)$ be two vertices of $G$ such that $xy \in E(G)$. Thus $H_1 = H + xy$ is a spanning subgraph of $G$. By (c) of Theorem A, it suffices to show that for any $e \in E(H_1)$, $H_1(e)$ is collapsible.

If $e \neq xy$, then $H_1(e)$ contains a 3-cycle $C_3$. Note that either every edge in $H_1(e)/C_3$ is in a 3-cycle or $G(e)/C_3$ has a 2-cycle $C_2$ such that every edge in $[G(e)/C_3]/C_2$ is in a $k$-cycle with $k \in \{2, 3\}$. It follows from (d) of Theorem A
that $G(e)$ is collapsible.

If $e = xy$, then since $n \geq 5$, $n - 2 \geq 3$. It follows from Theorem 11 of [3] that $G(e)$ is collapsible. □

**Lemma 2** Let $H$ be a graph that contains a nonspanning $\pi$-collapsible subgraph $K$, where $\pi = <V_1, V_2>$ is a bipartition of $V(H)$. If $e_\pi$ is contained in a collapsible subgraph of $H/\pi$, then the reduction of $H/\pi$ is the same as the reduction of $H$.

**Proof:** Let $H'$ and $(H/\pi)'$ denote the reductions of $H$ and $H/\pi$, respectively. Since $e_\pi$ is in a collapsible subgraph $L_1$ of $H/\pi$, the subgraph

$$L = H[(E(L_1) - \{e_\pi\}) \cup E(C_4)]$$

is a collapsible subgraph of $H$, by Theorem C. By the definition of contractions, we have

$$H/L \cong (H/\pi)/L_1.$$  \hspace{1cm} (1)

By (d) of Theorem A, (1) implies that $H' \cong (H/\pi)'$. □

By Theorem C and Lemma 1, it suffices to show that if $G$ is collapsible, then either (a) or (b) of Theorem 1 holds.

Note that the hypothesis of Theorem 1 implies that $n \geq 5$. Let $G'$ denote the reduction of $G/\pi$. If $G' = K_1$, then by (d) of Theorem A and by Theorem C, $G/\pi$ is collapsible and so (a) of Theorem 1 holds. Thus we assume that

$$G' \text{ is not collapsible.}$$  \hspace{1cm} (2)

Since contracting the edges does not increase the diameter and since the operation to get $G/\pi$ from $G$ does not increase the diameter either, we have $\text{diam}(G') \leq 2$. Thus by Theorem B, one of the conclusions of Theorem B must hold or $G' \cong K_2$.

Since $G$ is collapsible, if $e_\pi$ is in a collapsible subgraph of $G/\pi$, then by Lemma 2, $G'$ is collapsible, a contradiction. Hence we assume that

$$e_\pi \in E(G').$$  \hspace{1cm} (3)

If $G'$ does not have a cut edge, then by Theorem B, $G'$ is either a $K_{2,\ell}$, or an $S_{n,m}$, or the Petersen graph. In any case, by (3), one of the edges of $G'$ is $e_\pi$. It follows that the diameter of $G$ would be at least 3, contrary to the hypothesis of $\text{diam}(G) \leq 2$. 

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Since $G$ is collapsible, $G$ is 2-edge-connected. If $G'$ has a cut-edge, then this cut-edge must be created in the process of getting $e_\pi$. This, in conjunction with (3), implies that $e_\pi$ is the only cut-edge of $G'$. By Theorem B, we have

$$G' \cong K_2. \quad (4)$$

Let the $C_4$ subgraph be $C_4 = x_1x_2x_3x_4x_1$ and let $\pi = < V_1, V_2 >$ with $V_1 = \{x_1, x_3\}, V_2 = \{x_2, x_4\}$. Then by (4), $E(C_4)$ is an edge-cut of $G$. Let $G_1$ and $G_2$ denote the two sides of $G - E(C_4)$ such that $V_1 \subseteq V(G_1)$ and $V_2 \subseteq V(G_2)$. Since $\text{diam}(G) \leq 2$, at most one of the $V(G_i)$'s contains more than two vertices. Since $n \geq 5$, we may assume that

$$|V(G_1)| = 2, \quad |V(G_2)| = n - 2 \geq 3.$$ 

Since $\text{diam}(G) \leq 2$, for every vertex $w \in V(G_2) - \{x_2, x_4\}$, we must have $wx_2, wx_4 \in E(G)$. Thus $G$ has a spanning subgraph $H \cong K_{2,n-2}$ with $D_2(H) = \{x_2, x_4\}$. Since $H$ is not collapsible but $G$ is collapsible, and since $E(C_4)$ is an edge-cut of $G$, there must be vertices in $D_2(H)$ that are adjacent. Hence (b) of Theorem 1 holds. □

### THE PROOF OF THEOREM 2

In this section, for $v \in V(G)$, $N_G(v)$ denotes the set of vertices in $G$ that are adjacent to $v$.

Let $n = |V(G)|$. When $n \leq 4$, Theorem 2 is obvious. Thus we assume that $n \geq 5$. Let $G$ and $e$ satisfy the hypothesis of Theorem 2 and suppose that $G(e)$ is not collapsible. Denote $e = x_1y_1, x_1, y_1 \in V(G)$ and let $z_1$ denote $v(e)$. By contradiction, we assume that the conclusions of Theorem 2 is false. Let $G$ be a counterexample of Theorem 2 with as few vertices as possible. Thus, by the minimality of $G$, we have

$$G(e) \text{ is reduced.} \quad (5)$$

By (5) and by (b) of Theorem A,

$$G(e) \text{ is } K_3\text{-free.} \quad (6)$$

Lemma 3 $G(e)$ contains no 4-cycle $C$ with $E(C) \cap \{x_1z_1, z_1y_1\} = \emptyset$ (such a 4-cycle $C$ is called a forbidden 4-cycle).

Proof: Suppose that $G(e)$ has a forbidden 4-cycle $C$. Let $\pi$ denote the bipartition of $V(C)$. Since $x_1z_1, z_1y_1 \not\in E(C)$, we have

$$(G/\pi)(e) = G(e)/\pi. \quad (7)$$
Let \([((G/\pi)(e))']\) and \([G(e)/\pi]'\) denote the reductions of \((G/\pi)(e)\) and \(G(e)/\pi\), respectively. Then by (7)

\[
[((G/\pi)(e))'] = [G(e)/\pi]'.
\] (8)

By Theorem 1, either \(G/\pi\) is collapsible, or \(G\) has a spanning subgraph \(H \cong K_{2,n-2}\), with two vertices in \(D_2(H)\) adjacent in \(G\). If the latter case holds, then by Lemma 1, and by (c) of Theorem A, we have \([G(e)]' \cong K_1\) and so (a) of Theorem 2 holds, contrary to the assumption that \(G\) is a counterexample. Hence \(G/\pi\) must be collapsible.

Note that \(\text{diam}(G/\pi) \leq \text{diam}(G) \leq 2\). By the minimality of \(G\), \([G(e)/\pi]'\) must satisfy one of the conclusions of Theorem 2. If \(e_\pi \notin E([G(e)/\pi]')\), then \(e_\pi\) is in a collapsible subgraph of \([G(e)/\pi]'\). It follows by Lemma 2 that

\[
[G(e)]' = [G(e)/\pi]',
\] (9)

and so by (8) and (9), one of the conclusions of Theorem 2 must hold for \([G(e)]'\), contrary to the assumption that \(G\) is a counterexample. Hence we may assume that

\[
e_\pi \in E([G(e)/\pi]').
\] (10)

But then (10) and any one of (b), (c), (d), and (e) of Theorem 2 would imply that the diameter of \(G\) exceeds 2, a contradiction. This proves the lemma. \(\square\)

If \(\text{diam}(G(e)) \leq 2\), then by (5) and Theorem B, (a) or (b) or (c) of Theorem 2 must hold, contrary to the assumption that \(G\) is a counterexample. Hence by the hypothesis of \(\text{diam}(G) \leq 2\), we may assume that

\[
\text{diam}(G(e)) = 3.
\] (11)

By \(\text{diam}(G) \leq 2\) again, for any distinct vertices \(u, v \in V(G(e)) - \{z_1\}\), either

\[
d_{G(e)}(u, v) \leq 2,
\] (12)

or in \(G(e)\),

all shortest \((u,v)\)-paths are of length 3 and contain \(x_1z_1, y_1z_1\). (13)

By (11), there are distinct vertices \(x, y \in v(G)\) such that \(d_{G(e)}(x, y) = 3\). By (13), we may assume that \(y = y_1\) and that \(xz_1z_1y\) is a shortest \((x,y)\)-path.

By the hypothesis that \(G\) is collapsible, we have \(\kappa'(G) \geq 2\) and so

\[
\kappa'(G(e)) \geq 2.
\] (14)

This, in conjunction with (5) and (b) of Theorem A, implies

\[
2 \leq \delta(G(e)) \leq 3.
\]
Thus we have
\[ 2 \leq \delta(G) \leq 3. \] (15)

**Lemma 4** \( \kappa(G(e)) \geq 2. \)

**Proof:** By contradiction, we assume that \( G(e) \) has a cut-vertex \( v \) and so it has two nontrivial connected subgraphs \( G_1 \) and \( G_2 \) such that
\[ E(G(e)) = E(G_1) \cup E(G_2) \text{ and } V(G_1) \cap V(G_2) = \{v\}. \]
By (11), we may assume that in \( G_2 \), there is a vertex \( u \) such that
\[ d_{G(e)}(v, u) \geq 2. \] (16)
Thus every vertex in \( V(G_1) - \{v\} \) is adjacent to \( v \), by (11). By (5) and by (a) of Theorem A, both \( G_1 \) and \( G_2 \) are reduced. It follows that \( G_1 \) must have a cut-edge, contrary to (14). \( \Box \)

By Menger’s Theorem ([1], page 16), by Lemma 4 and (11), we have

every two edges of \( G(e) \) is in a cycle of length at most 6. (17)

We shall divide the rest of the proof into several cases.

**Case 1** Either \( \deg(x_1) \) or \( \deg(y) \) is equal to \( \delta(G(e)) \).

(1A) \( \deg(y) = 2. \)
Let \( z, x_1 \) be the two vertices adjacent to \( y \) in \( G(e) \). By (11) and (17), we may assume that \( G(e) \) has a 6-cycle
\[ H_1 = yz_1x_1xz_2zy. \]
Since \( \text{diam}(G) \leq 2 \) and by \( \deg(y) = 2, \) for every vertex \( w \in V(G(e)) - \{y, z, x_1, x_1\} \)
\[ \text{either } zw \text{ or } x_1w \text{ is in } E(G(e)). \] (18)
By (6) and (18), and by Lemma 3,
\[ \deg(x_2) = \deg(x) = 2. \] (19)
Since \( G \) is collapsible, \( G \neq H_1. \) But by (14), (18) and (19), for every \( w \in V(G) - V(H_1), \) we must have
\[ wz, wx_1 \in E(G(e)). \] (20)
By (6) and Lemma 3, \( E(G(e)) \) consists of edges in \( E(H_1) \) and edges described in (20) only. It follows that \( G \) is a subdivision of \( K_t, \) for some \( t \geq 2, \) and so \( G \)
is not collapsible, a contradiction.

(1B) \( \deg(y) = \delta(G) = 3. \)

Let \( z_1, z_2, z_3 \) be the vertices adjacent to \( y \) in \( G(e) \). By (5), (6) and (17), we may assume that there are two more vertices \( z_4, z_5 \) such that in \( G(e) \),

\[
yz_2, z_2 z_4, z_4 x, yz_3, z_3 z_5, z_5 x \in E(G(e)).
\]

By \( \delta(G) = 3 \), and by (6), (12) and (13), we can find \( z_6, z_7 \) so that

\[
z_2 z_7, z_7 z_5, z_3 z_6, z_6 z_4 \in E(G(e)).
\]

Let \( H_2 \) denote the induced subgraph of \( G(e) \) with

\[
V(H_2) = \{y, x_1, x, x_1, z_2, z_3, z_4, z_5, z_6, z_7\}.
\]

By (5), (6) and Lemma 3, \( E(H_2) \) consists of the edges described in (21) and (22), together with \( \{yz_1, x_1 z_1, x_1 x\} \). Since \( \delta(G) = \deg(y) = 3 \), for every \( w \in V(G(e)) - V(H_2) \),

one of \( wz_1, wz_2, wz_3 \) is in \( E(G(e)) \). \( \tag{23} \)

Since \( H_2/z_1y \) is a subgraph of the Petersen graph, it is not collapsible, and so there must be a vertex \( w_1 \in V(G(e)) - V(H_2) \).

Claim 1 If \( w_1 x_1 \in E(G(e)) \), then \( w_1 z_6, w_1 z_7 \in E(G(e)) \).

Suppose that \( w_1 z_6 \notin E(G(e)) \). By (12) and (13), \( d_{G(e)}(w_1, z_6) = 2 \) and so there must be some vertex \( w_2 \in V(G(e)) \) such that

\[
w_1 w_2, w_2 z_6 \in E(G(e)).
\]

Then by (23), either \( w_2 x_1 \), or \( w_2 z_2 \), or \( w_2 z_3 \) is an edge of \( G(e) \). By (6), it must be \( w_2 z_2 \in E(G) \). By \( \delta(G) = 3 \), there is some \( w_3 \in V(G) - [V(H_2) \cup \{w_1, w_2\}] \) such that \( w_3 w_1 \in E(G) \). By (6), \( w_3 \notin V(H_2) \cup \{w_1, w_2\} \). By (23), \( G(e) \) has either a \( k \)-cycle, \( 2 \leq k \leq 3 \), or a forbidden 4-cycle, contrary to (5) or Lemma 3.

Similarly, \( w_1 z_7 \) is in \( E(G(e)) \). \( \square \)

Claim 2 The degrees of \( z_4, z_5, x, x_1 \) in \( G(e) \) are at most 3.

It follows from Claim 1 and Lemma 3 that \( \deg_{G(e)}(x_1) \) is at most 3.

Suppose that there is a vertex \( w \notin V(H_2) \) such that \( wz_4 \in E(G(e)) \). By (23), one of \( wz_1, wz_2, wz_3 \) is in \( E(G(e)) \). But then by Claim 1 again, \( G(e) \) contains either a \( K_3 \) or a forbidden 4-cycle, contrary to either (6) or Lemma 3.

Similarly, \( z_6, x \) must have degree 3 also. \( \square \)

By Claim 2, we have \( G \cong G(e)/z_1 x_1 \) is isomorphic to the Petersen graph and so \( G \) is not collapsible, a contradiction.
(1C) \( \deg_{G(e)}(x_1) = \delta(G) \).

If \( y \) is adjacent to a vertex \( w \) such that
\[
d_{G(e)}(w, x_1) = 3,
\]
then we are back to Case 1A or Case 1B, by renaming the vertices \( w, y, z_1, x_1 \)
by \( x, x_1, z_1, y \), respectively. Hence we may assume that
\[
\text{for any } w \notin \{y, z_1, x_1, x\}, \ wy \in E(G(e)) \implies wx_1 \in E(G(e)). \tag{24}
\]
By (24), we have \( \deg_{G(e)}(y) = \deg_{G(e)}(x_1) = \delta(G) \), and so we are back to Case
1A or Case 1B again.

**Case 2** \( \deg_{G(e)}(x) = \delta(G) \) and \( \deg_{G(e)}(x_1) > \delta(G) \), \( \deg_{G(e)}(y) > \delta(G) \).

(2A) \( \delta(G) = 2 \).

Let \( x_1, x_2 \) be the vertices in \( G(e) \) adjacent to \( x \). By (17) and by the assumption
of \( d_{G(e)}(x, y) = 3 \), there is some \( z \in V(G(e)) \) such that
\[
H_3 = yzx_2x_1x_1y
\]
is a 6-cycle of \( G(e) \). Since \( \deg_{G(e)}(x_1) \geq 3 \), there is some vertex \( w_1 \in V(G(e)) - V(H_3) \) such that \( w_1x_1 \in E(G(e)) \). By (12) and (13), we must have
\[
d_{G(e)}(w_1, x_2) \leq 2.
\]
By Lemma 3, there must be a vertex \( w_2 \notin V(H_3) \cup \{w_1\} \) with \( w_1w_2, w_2x_2 \in E(G(e)) \). By (12), (13) and Lemma 3, there are must be a vertex \( w_3 \notin V(H_3) \cup \{w_1, w_2\} \) with \( w_2w_3, w_3y \in E(G(e)) \). Then by \( N_{G(e)}(z) = \{x_1, x_2\} \), and by (12) and (13), either \( w_3x_1 \) or \( w_3x_2 \) is in \( E(G(e)) \). It follows that \( G(e) \)
contains either a forbidden 4-cycle or a 3-cycle, contrary to Lemma 3 or to (6).

(2B) \( \delta(G) = 3 \).

Let \( x_1, x_2, x_3 \) be the vertices in \( G(e) \) adjacent to \( x \). By (17), by the assumption
of \( d_{G(e)}(x, y) = 3 \) and by Lemma 3, there are vertices \( x_4, x_5 \) such that
\[
x_2x_4, x_4x_5, x_5x_6, x_6y \in E(G(e)).
\]
By (12), (13), we have \( \deg_{G(e)}(x_3, x_4) = 2 \), and so by Lemma 3, there is a vertex
\( x_6 \notin \{y, \ x, x_1, x_2, x_3, x_4, x_5\} \) with
\[
x_6x_4, x_3x_6 \in E(G(e)).
\]
By (6) and by \( \deg_{G(e)}(y) \geq 4 \), there is a vertex \( x_7 \notin \{x_1, x_2, x_3, x_4, x_5, x_6, x, y, z_1\} \)
with \( x_7y \in E(G(e)) \). By (12) and (13), we have \( d_{G(e)}(x_7, x) = 2 \) and so by
\( N_{G(e)}(x) = \{x_1, x_2, x_3\} \), either \( x_1x_7 \), or \( x_2x_7 \) or \( x_3x_7 \) is in \( E(G(e)) \). It follows
by Lemma 3 that $x_1x_7 \in E(G(e))$. Similarly, $d_{G(e)}(x_6, x_7) = 2$ and so there is some vertex $w$ with $wx_6,(wx_7 \in E(G(e))$. Then by $N_{G(e)} = \{x_1, x_2, x_3\}$, and by (12) and (13), one of $x_1w, x_2w, x_3w$ is in $E(G(e))$. It follows that $G(e)$ has either a $K_3$ or a forbidden 4-cycle, contrary to either (6) or Lemma 3.

**Case 3** All $deg_{G(e)}(x_1), deg_{G(e)}(y), deg_{G(e)}(z)$ are greater than $\delta(G)$.
Let $z \in V(G(e)) - \{x_1\}$ be a vertex with $deg_{G(e)}(z) = \delta(G)$.

(3A) $zy \in E(G(e))$.
If $d_{G(e)}(z, x_1) = 3$, then by (13), $zy \in E(G(e))$, and so we are back to Case 2 with the path $zyz_1x_1$ replacing $x_1x_1y$. Hence $d_{G(e)}(z, x_1) \leq 2$. If $zx_1 \in E(G(e))$, then by replacing $x_1x_1y$ by $x_1x_1y$, we are back to Case 2 again. Thus we assume that there is some $z_2 \in V(G(e))$ with $x_2x_2x_1 \in E(G(e))$. If $deg_{G(e)}(x) = 2$, then by (12), $d_{G(e)}(z, x) = 2$ and so by (6) and $N_{G(e)}(x) = \{y, x_2\}$, we must have $zy \in E(G(e))$, contrary to the assumption of $deg_{G(e)}(z, y) = 3$. Thus we assume that

$$deg_{G(e)}(z) = 3.$$

Let $y, x_2, x_3$ be the vertices in $G(e)$ that are adjacent to $z$ and that $zx_2, x_2x_1$ are in $E(G(e))$. By (12) and (6), and by $zy \not\in E(G(e))$, we have $zx_3 \in E(G(e))$. By $deg_{G(e)}(x) \geq 4$, we assume that $x_4, x_5$ are in $V(G(e))$ with $x_4x, x_5x \in E(G(e))$. By (12) with $\{u, v\} = \{z, x_i\}, (i = 1, 2)$, and by (6), we must have $x_4y, x_5y \in E(G(e))$. It follows that $G(e)$ contains a forbidden 4-cycle, contrary to Lemma 3.

(3B) $zx_1 \in E(G(e))$.
Since $deg_{G(e)}(z) \geq deg_{G(e)}(x) + 1$, by (12) with $\{u, v\}$ being $z$ and one vertex adjacent to $x$, $G(e)$ contains either a 3-cycle or a forbidden 4-cycle, contrary to (6) or to Lemma 3.

(3C) $zx \in E(G(e))$.
Since $deg_{G(e)}(y) \geq deg_{G(e)}(x) + 1$, by (12) and (13) with $u, v$ being nd one vertex adjacent to $y$, $G(e)$ must contain a 3-cycle or a forbidden 4-cycle, contrary to (6) or Lemma 3.

(3D) $zx, zx_1, zy \not\in E(G(e))$.
By (12) and the assumption of Case 3D, there are $x_2, x_3, x_4$ such that

$$zx_2, x_2x, x_3x, x_3x_1, zx_4, x_4y \in E(G(e)).$$

Hence $deg_{G(e)}(z) = 3$ and so $deg_{G(e)}(z) \geq 4$. By (12), (6) and by $N_{G(e)}(z) = \{x_2, x_3, x_4\}$, there is a vertex adjacent to $z$ and one of $x_3, x_4$. By $deg_{G(e)}(x) \geq 4$, by (12) and by Lemma 3, we may assume that $x_5$ is adjacent to both $z$ and $x_3$. By (12) and Lemma 3 again, there is a vertex $x_6$ adjacent to both $y$ and $x_2$. By the same reason once more, there is some vertex $w$ adjacent to $x_5$ and $x_6$. 341
Since \( N_{G(e)}(x) = \{x_2, x_3, x_4\} \), and by (12), one of \( wx_2, wx_3, wx_4 \) is an edge of \( G(e) \). It follows that \( G(e) \) has either a 3-cycle or a forbidden 4-cycle, contrary to (6) or to Lemma 3.

Since all cases lead to contradictions, the proof of Theorem 2 is complete.

AN APPLICATION

We conclude this note with an application of Theorem 2. The line graph of \( G \), denoted by \( L(G) \), has \( E(G) \) as its vertex set, where two vertices in \( L(G) \) are adjacent in \( L(G) \) if and only if the corresponding edges are adjacent in \( G \). A trail \( T \) of \( G \) is called a dominating trail of \( G \) if \( G - V(T) \) is edgeless.

Theorem D (Harary and Nash-Williams [5]) Let \( G \) be a graph with at least 3 edges. \( L(G) \) has a Hamilton cycle if and only if \( G \) has a dominating eularian subgraph. □

Imitating the proof of Theorem D, one has:

**Lemma 4** Let \( e \) be an edge of \( G \). Then \( L(G) \) has a Hamilton path starting with \( e \) if and only if \( G \) has a dominating trail starting with \( e \). □

A dominating trail of \( G \) starting with an edge \( e \in E(G) \) is called a dominating \( e \)-trail of \( G \).

**Lemma 5** ([7], Corollary 9) If \( G \) is collapsible, then for any \( v, u \in V(G) \), (possibly \( u = v \)), there is a spanning \((v, u)\)-trail in \( G \). □

**Corollary 1** If \( G \) is a graph of diameter at most two, then for any edge \( e \in E(G) \), in \( L(G) \), the line graph of \( G \), has a Hamilton path starting with \( e \).

**Proof:** Let \( e \) be an edge of \( G \). To avoid trivial cases, we assume that \( G \) has at least 3 edges. By Lemma 4, it suffices to show that \( G \) has a dominating \( e \)-trail.

**Case 1** \( G \) is collapsible.

Note that \( G \) has an \( e \)-trail if and only if \( G(e) \), the graph obtained from \( G \) by subdividing the edge \( e \) once, has a dominating trail with \( v(e) \) at an end of the trail, (call such trails dominating \( v(e) \)-trails). Note that any spanning \( v(e) \)-trail (spanning trail starting with \( v(e) \)) is a dominating \( v(e) \)-trail.

If \( G(e) \) is collapsible, then by Lemma 5, \( G(e) \) has a \( v(e) \)-trail and we are done. So suppose that \( G(e) \) is not collapsible. Let \( [G(e)]' \) denote the reduction of \( G(e) \). By Theorem 2, \([G(e)]' \in \{K_{2,t}, S_{1,m}, K_{2,t}, S_{1,m}\}, (t \geq 2, l \geq 2, m \geq 1)\). Since \( G \) is collapsible, \( v(e) \) must be a trivial vertex of \([G(e)]' \). It is then easy to check
that \([G(e)]'\) has a spanning \(v(e)\)-trail. Thus \(G\) has a spanning trail starting with \(v(e)\) and so \(G\) has a dominating \(e\)-trail.

**Case 2** \(G\) is not collapsible.

Let \(G'\) denote the reduction of \(G\). By Theorem B, \(G' \in \{K_2, K_{1,t}, K_{2,t}, S_{l,m}, P\}\), where \(t \geq 2, l, m \geq 1\) and \(P\) is the Petersen graph.

If \(G' = P\), then by Corollary 7 of [7], \(G = G' = P\) and so \(G\) has a spanning \(e\)-trail. By \(
\text{diam}(G) \leq 2\), if \(G' = K_{1,t}\), then \(G = G' = K_{1,t}\) again and so \(G\) has a dominating \(e\)-trail. Hence we assume \(G' \in \{K_2, K_{2,t}, S_{l,m}\}\). Since \(
\text{diam}(G) \leq 2\), at most one vertex of \(G'\) is nontrivial, and if \(H\) denotes the only nontrivial collapsible subgraph of \(G\), then \(H\) is spanned by \(K_{1,t'}\), \((t' \geq 1)\). Thus \(G\) has a spanning \(e\)-trail again.

This proves Corollary 1. □

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