The reduction of graph families closed under contraction

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Received 16 March 1994; revised 24 January 1995

Abstract

Let \( \mathcal{S} \) be a family of graphs. Suppose there is a nontrivial graph \( H \) such that for any supergraph \( G \) of \( H \), \( G \) is in \( \mathcal{S} \) if and only if the contraction \( G/H \) is in \( \mathcal{S} \). Examples of such an \( \mathcal{S} \): graphs with a spanning closed trail; graphs with at least \( k \) edge-disjoint spanning trees; and \( k \)-edge-connected graphs (\( k \) fixed). We give a reduction method using contractions to find when a given graph is in \( \mathcal{S} \) and to study its structure if it is not in \( \mathcal{S} \). This reduction method generalizes known special cases.

Keywords: Contraction; Spanning tree; Edge-arboricity; Edge-connectivity; Eulerian; Super-eulerian

1. Introduction

We use the notation of Bondy and Murty [1], except that we do not allow graphs to have loops, we regard \( K_1 \) as \( k \)-edge-connected for all \( k \in \mathbb{N} \), and we call a graph trivial if it is edgeless.

Let \( H \) (not necessarily connected) be a subgraph of \( G \). The contraction \( G/H \) is the graph obtained from \( G \) by contracting all edges in \( H \) and by deleting any resulting loops. If \( e \in E(G) \), then we denote \( G/G[e] \) by \( G/e \).

A collection \( \mathcal{S} \) of graphs is called a graph family or a family. When \( G \) and \( H \) are graphs, if \( H \) is a subgraph of \( G \), we denote this by \( H \subseteq G \). Call a family \( \mathcal{S} \) of graphs closed under contraction if

\[
G \in \mathcal{S}, \ e \in E(G) \Rightarrow G/e \in \mathcal{S}.
\]  

(1)

Call a family \( \mathcal{C} \) of graphs complete if \( \mathcal{C} \) satisfies these three axioms:

(C1) \( \mathcal{C} \) contains all edgeless graphs;
(C2) \( \mathcal{C} \) is closed under contraction;
(C3) \( H \subseteq G, \ H \in \mathcal{C}, \ G/H \in \mathcal{C} \Rightarrow G \in \mathcal{C} \).

\(^{\dagger}\) Sadly, the author passed away on April 20, 1995.
Call a family $\mathcal{F}$ of graphs free if these three axioms hold:

(F1) $\mathcal{F}$ contains all edgeless graphs;
(F2) $G \in \mathcal{F}, H \subseteq G \Rightarrow H \in \mathcal{F}$;
(F3) For any induced subgraph $H$ of $G$,

$$H \in \mathcal{F} \quad \text{and} \quad G/H \in \mathcal{F} \Rightarrow G \in \mathcal{F}.$$ 

For any family $\mathcal{I}$ of graphs, we define the kernel $\mathcal{I}^O$ of $\mathcal{I}$ to be the family

$$\mathcal{I}^O = \{ H \mid \text{for every supergraph } G \text{ of } H, G \in \mathcal{I} \Leftrightarrow G/H \in \mathcal{I} \}. \quad (2)$$ 

Obviously, $\mathcal{I}^O$ contains all edgeless graphs. If $\mathcal{I}^O = \{\text{edgeless graphs}\}$, then we call $\mathcal{I}^O$ trivial.

Let $\mathcal{I}$ be a family $\mathcal{I}$ with a nontrivial kernel $\mathcal{I}^O$ that is closed under contraction. Is a given graph $G$ (say) in $\mathcal{I}$? Subgraphs of $G$ in the kernel $\mathcal{I}^O$ can each be contracted, and this can be repeated, until a ‘reduced’ graph $G_1$ (say) is obtained, having no nontrivial subgraph in $\mathcal{I}^O$, where (2) implies

$$G \in \mathcal{I} \quad \text{if and only if} \quad G_1 \in \mathcal{I}. \quad (3)$$

By (3), to know if $G \in \mathcal{I}$ it suffices merely to know if the ‘reduced’ graph $G_1$ is in $\mathcal{I}$. If $\mathcal{I}^O$ is nontrivial, then this can be easier than determining directly whether $G \in \mathcal{I}$. (We shall prove that this ‘reduced graph’ $G_1$ is uniquely determined by $G$ and $\mathcal{I}^O$, if $\mathcal{I}^O$ is closed under contraction; that the family of all such ‘reduced’ graphs, corresponding to a given $\mathcal{I}$, is free; that if $\mathcal{I}$ or $\mathcal{I}^O$ is closed under contraction, then $\mathcal{I}^O$ is a complete family; that all complete families arise as kernels; and that all free families arise as families of ‘reduced graphs’.)

For any family $\mathcal{F}$ of graphs, define

$$\mathcal{F}^R = \{ G \mid G \text{ has no nontrivial subgraph in } \mathcal{F} \} \quad (4)$$

and

$$\mathcal{F}^C = \{ G \mid G \text{ has no nontrivial contraction in } \mathcal{F} \}.$$ 

(This family $\mathcal{F}^R$ is a family of ‘reduced’ graphs corresponding to $\mathcal{F}$, when $\mathcal{F}$ is a kernel. The family $\mathcal{F}^C$ is the dual concept.) We shall also show that if $\mathcal{C}$ and $\mathcal{F}$ are families of graphs such that $\mathcal{C}^R = \mathcal{F}$ and $\mathcal{F}^C = \mathcal{C}$, then $\mathcal{C}$ is a complete family and $\mathcal{F}$ is a free family. Furthermore, all complete and free families arise this way.

2. Examples: complete families and kernels

Define the family $\mathcal{PL}$ of supereulerian graphs: $G \in \mathcal{PL}$ whenever $G$ has a spanning closed trail, and $K_1$ is regarded as being in $\mathcal{PL}$. Thus, if $G \in \mathcal{PL}$ then $G$ is the spanning supergraph of an eulerian graph, and $K_1$ is regarded as eulerian. Clearly, $\mathcal{PL}$ is closed under contraction. A graph $G$ is called collapsible if for every even
subset $X$ of $V(G)$, $G$ has a spanning connected subgraph $H$ with $X$ as its set of odd-degree vertices (see [2,3]). By Theorem 3 of [2] and its corollary, the family $\mathcal{CL}$ of graphs whose components are collapsible is a complete family, and $\mathcal{CL} \subseteq \mathcal{PL}^O$. We conjecture that $\mathcal{CL} = \mathcal{PL}^O$.

For any natural number $k$, let $\mathcal{G}(k)$ be the family of graphs with the property that for any $2k$ vertices $s_1, t_1, s_2, t_2, \ldots, s_k, t_k \in V(G)$ (not necessarily distinct) there are pairwise disjoint $(s_i, t_i)$-paths $P_i$ $(1 \leq i \leq k)$. The family $\mathcal{G}(k)$ is easily shown to be complete, and its members are called weakly $k$-linked. Seymour [7] and Thomassen [8] have characterized $\mathcal{G}(2)$.

Lai [4] (and Theorem 4 of [5]) proved that if $\mathcal{F}$ is a complete family and if $\mathcal{G}_k$ is the family of graphs at most $k$ edges short of being in $\mathcal{F}$, then $\mathcal{G}_k^O = \mathcal{G}$.

3. Complete families and kernels

In the results of this section, $\mathcal{F}$, $\mathcal{I}$ and $\mathcal{C}$ will be various graph families, and $\mathcal{G}$ will often be complete. For the special case $\mathcal{F} = \mathcal{PL}$ and $\mathcal{G} = \mathcal{CL}$, some results below were first done in [2]: Theorem 4, Corollary 2 of Theorem 4, and Lemma 4 of [2] are generalized below to Theorem 3.7, Corollary 3.8, and Lemma 3.9, respectively.

Lemma 3.1. Let $\mathcal{F}$ be a graph family. If

$$\mathcal{F} \text{ contains all edgeless graphs,}$$

then $\mathcal{F}^O \subseteq \mathcal{F}$.

Proof. Let $\mathcal{F}$ be a family satisfying (5) and suppose $G' \in \mathcal{F}^O$. By (2),

$$G \in \mathcal{F} \iff G\!\setminus\!G' \in \mathcal{F}$$

holds for every supergraph $G$ of $G'$. Set $G = G'$ in (6) and use (5) to get $G' \in \mathcal{F}$. Hence, $\mathcal{F}^O \subseteq \mathcal{F}$. \qedsymbol

Lemma 3.2. If $\mathcal{F}$ is a graph family then $(\mathcal{F}^O)^O = \mathcal{F}^O$; also, all edgeless graph are in $\mathcal{F}$ if and only if $\mathcal{F}^O \subseteq \mathcal{F}$.

Proof. Let $\mathcal{F}$ be a graph family. Now, all edgeless graphs are in $\mathcal{F}^O$, and so $\mathcal{F}^O \subseteq \mathcal{F}$ implies that $\mathcal{F}$ contains all edgeless graphs. Set $\mathcal{F} = \mathcal{F}$ in Lemma 3.1 to get the last part of Lemma 3.2. Set $\mathcal{F}^O = \mathcal{F}$ in Lemma 3.1 to get $(\mathcal{F}^O)^O \subseteq \mathcal{F}^O$. It remains to prove

$$\mathcal{F}^O \subseteq (\mathcal{F}^O)^O.$$  

Let $H \in \mathcal{F}^O$, let $G'$ be a supergraph of $H$, and let $G$ be an arbitrary supergraph of $G'$. Hence,

$$G\!\setminus\!G' = (G\!\setminus\!H)/(G'\!\setminus\!H),$$

(8)
and since \( H \in \mathcal{F}^0 \), (2) implies

\[
G/H \in \mathcal{F} \iff G \in \mathcal{F}.
\]  

(9)

If \( G' \in \mathcal{F}^0 \), then by (2),

\[
G \in \mathcal{F} \iff G/G' \in \mathcal{F},
\]  

(10)

and by (8)–(10),

\[
G/H \in \mathcal{F} \iff G/G' \in \mathcal{F} \iff (G/H)/(G'/H) \in \mathcal{F}.
\]  

(11)

Since \( G/H \) can be any supergraph of \( G'/H \), (11) implies \( G'/H \in \mathcal{F}^0 \).

Conversely, if \( G' \notin \mathcal{F}^0 \), then for some supergraph \( G \) of \( G' \),

\[
G \in \mathcal{F} \nleftrightarrow G/G' \in \mathcal{F},
\]  

(12)

and so by (9), (12), and (8),

\[
G/H \in \mathcal{F} \nleftrightarrow (G/H)/(G'/H) \in \mathcal{F}.
\]  

(13)

Therefore, (2) implies that \( G'/H \notin \mathcal{F}^0 \).

By the last two paragraphs,

\[
G' \in \mathcal{F}^0 \iff G'/H \in \mathcal{F}^0,
\]

when \( G' \) is an arbitrary supergraph of \( H \). Hence, \( H \in (\mathcal{F}^0)^0 \), whence (2) implies (7). \( \square \)

**Theorem 3.3.** For any graph family \( \mathcal{F} \), if \( \mathcal{F} \) or \( \mathcal{F}^0 \) is closed under contraction, then \( \mathcal{F}^0 \) is complete.

**Proof.** Let \( \mathcal{F} \) be a graph family.

First we show that \( \mathcal{C} = \mathcal{F}^0 \) satisfies (C1) and (C3). By Lemma 3.2, \((\mathcal{F}^0)^0 = \mathcal{F}^0\). This and Lemma 3.2 imply that \( \mathcal{F}^0 \) satisfies (C1). Also, \((\mathcal{F}^0)^0 = \mathcal{F}^0\) implies that \( \mathcal{C} = \mathcal{F}^0 \) satisfies (C3): for if \( H \in \mathcal{F}^0 \) and \( H \subseteq G \) then \( H \in (\mathcal{F}^0)^0 \) and so (2) gives \( G/H \in \mathcal{F}^0 \Rightarrow G \in \mathcal{F}^0 \).

By hypothesis, either \( \mathcal{F} \) or \( \mathcal{F}^0 \) is closed under contraction. In the latter case \( \mathcal{F}^0 \) satisfies (C2), and so \( \mathcal{F}^0 \) is complete.

It only remains to suppose that \( \mathcal{F} \) is closed under contraction and to prove that \( \mathcal{F}^0 \) is closed under contraction. Let \( G \in \mathcal{F}^0 \). For all supergraphs \( G' \) of \( G \), (2) implies

\[
G' \in \mathcal{F} \iff G'/G \in \mathcal{F},
\]  

(14)

For any edge \( e \in E(G) \), we have

\[
(G'/e)/(G/e) = G'/G.
\]  

(15)
To prove that \( \mathcal{F}^O \) is closed under contraction, it suffices to prove \( G/e \in \mathcal{F}^O \), i.e., by (2), that
\[
G/e \in \mathcal{F} \Leftrightarrow (G/e)/(G/e) \in \mathcal{F}
\] (16)
for all supergraphs \( G/e \) of \( G/e \). Let \( G' \) be any supergraph of \( G \).
Suppose that \( G' \in \mathcal{F} \). Since \( \mathcal{F} \) is closed under contraction,
\[
G'/e \in \mathcal{F}
\] (17)
and
\[
G'/G \in \mathcal{F}.
\] (18)
By (18) and (15),
\[
(G'/e)/(G/e) \in \mathcal{F}.
\] (19)
Suppose that \( G' \notin \mathcal{F} \). By (14), we have \( G'/G \notin \mathcal{F} \), and so by (15),
\[
(G'/e)/(G/e) \notin \mathcal{F}.
\] (20)
By (20) and since \( \mathcal{F} \) is closed under contraction,
\[
G'/e \notin \mathcal{F}.
\] (21)
When \( G' \in \mathcal{F} \), both (17) and (19) hold, but if \( G' \notin \mathcal{F} \), then both (21) and (20) hold. Therefore, (16) holds, as claimed. □

**Theorem 3.4.** For any family \( \mathcal{C} \) of graphs that is closed under contraction, these are equivalent:
(a) \( \mathcal{C} \) is the kernel of some graph family closed under contraction;
(b) \( \mathcal{C} \) is a complete family;
(c) \( \mathcal{C} = \mathcal{C}^O \).

**Proof.** (a) ⇒ (b): By Theorem 3.3.
(b) ⇒ (c): By (b), \( \mathcal{C} \) is a complete family, and so (C1) and Lemma 3.1 give \( \mathcal{C}^O \subseteq \mathcal{C} \). Now suppose that \( H \in \mathcal{C} \), and let \( G \) satisfy \( H \subseteq G \). Since \( \mathcal{C} \) is complete, \( G/H \in \mathcal{C} \Leftrightarrow G \in \mathcal{C} \), because axiom (C2) implies ‘\( \leftarrow \)’ and axiom (C3) implies ‘\( \Rightarrow \)’. Hence, \( H \in \mathcal{C}^O \), and (c) follows.
(c) ⇒ (a): If (c) holds, then \( \mathcal{C} \) is the kernel of itself. □

Hong-Jian Lai (personal communication) has shown that part (a) of Theorem 3.4 can be replaced by ‘\( \mathcal{C} \) is the kernel of some graph family that is both closed under contraction and not complete’.

Let \( \mathcal{F} \) be the family of all connected graphs of odd order. Then \( \mathcal{F} = \mathcal{F}^O \), and since \( \mathcal{F} \) is not closed under contraction, neither is \( \mathcal{F}^O \). Therefore, the kernel \( \mathcal{F}^O \) is
not complete. Hence, in Theorems 3.3 and 3.4, we need the hypothesis of closure under contraction.

By (a) ⇔ (c) of Theorem 3.4, any kernel \( \mathcal{C} \) of a graph family closed under contraction satisfies (C2), and hence contains multigraphs of order 2. For practical purposes, to test whether a graph family \( \mathcal{F} \) (closed under contraction) has a nontrivial kernel \( \mathcal{F}^0 \), simply look for an order 2 multigraph \( H \) in \( \mathcal{F}^0 \) of (2). This is generally easy to check.

A family \( \mathcal{F} \) of graphs is called closed under edge-addition if for any graph \( G \) and edge \( e \in E(G) \), \( G - e \in \mathcal{F} \) implies \( G \in \mathcal{F} \).

**Theorem 3.5.** In any complete family, the subfamily of connected graphs is closed under edge-addition.

**Proof.** Let \( \mathcal{C} \) be the subfamily of connected graphs in a complete family, let \( G \) be a graph and let \( e \in E(G) \). Suppose \( G - e \in \mathcal{C} \). By (b) ⇒ (c) of Theorem 3.4, \( G - e \in \mathcal{C}^0 \), and so \( G \in \mathcal{C} \Leftrightarrow G/(G - e) \in \mathcal{C} \). Since \( G - e \) is connected and \( \mathcal{C} \) is complete, \( G/(G - e) = K_1 \in \mathcal{C} \). Hence \( G \in \mathcal{C} \). \( \square \)

**Lemma 3.6.** If \( \mathcal{C} \) is complete and \( G \in \mathcal{C} \), then \( G \cup K_1 \in \mathcal{C} \).

**Proof.** Apply (C3) with \( H \subseteq G \) of (C3) replaced by \( G \subseteq G \cup K_1 \). Then \( G/H \) of (C3) is an edgeless graph, and by (C1) it is in \( \mathcal{C} \). \( \square \)

**Theorem 3.7.** Let \( \mathcal{C} \) be a complete family of graphs. Let \( H \) be a graph containing subgraphs \( H_1 \) and \( H_2 \), and satisfying

\[
H_1 \cup H_2 = H.
\]

If \( H_1, H_2 \in \mathcal{C} \), then \( H \in \mathcal{C} \).

**Proof.** Let \( H \) be a graph with subgraphs \( H_1 \) and \( H_2 \) satisfying (22). Suppose that \( \mathcal{C} \) is a complete graph family, and suppose \( H_1, H_2 \in \mathcal{C} \).

The graph \( H/H_1 \) can be obtained from \( H_2 \) by a sequence of edge-additions, additions of isolated vertices, and contractions (contract newly added edges, to identify certain vertices of \( H_2 \) in \( H \)). Since \( H_2 \in \mathcal{C} \) and since \( \mathcal{C} \) is complete, \( H/H_1 \in \mathcal{C} \), by (C2), by Theorem 3.5, and by Lemma 3.6.

Since \( \mathcal{C} \) is complete, (b) ⇒ (c) of Theorem 3.4 implies \( H_1 \in \mathcal{C} = \mathcal{C}^0 \). Hence \( H \in \mathcal{C} \), because (2) implies

\[
H \in \mathcal{C} \Leftrightarrow H/H_1 \in \mathcal{C} \]. \( \square \)

**Corollary 3.8.** Let \( \mathcal{C} \) be a complete family and let \( G \) be a graph. Let \( E'' \) be a minimal edge set such that every component of \( G - E'' \) is in \( \mathcal{C} \). Let \( E' \) be the edges of \( G \) that lie in no subgraph of \( G \) in \( \mathcal{C} \). Then \( E'' = E' \) and the set of maximal subgraphs of \( G \) in \( \mathcal{C} \) is unique.
Proof. If $e \in E(G) - E''$ then $e \notin E'$, and so $E' \subseteq E''$. By contradiction, suppose that there is an edge $xy \in E'' - E'$. Let $H_x$ and $H_y$ denote the components of $G - E''$ containing $x$ and $y$, respectively. Thus, $H_x, H_y \in \mathcal{C}$. Since $xy \notin E'$, $xy$ is in a subgraph $H_{xy}$ (say) in $\mathcal{C}$. By Theorem 3.7, $H_x \cup H_{xy} \in \mathcal{C}$ and so $(H_x \cup H_{xy}) \cup H_y \in \mathcal{C}$. Therefore, each component of $G - (E'' - E(H_{xy}))$ is in $\mathcal{C}$, contrary to the minimality of $E''$. Hence, $E''$ is uniquely determined. Since the maximal connected subgraphs of $G$ in $\mathcal{C}$ are the components of $G - E''$, they are uniquely determined, too. \qed

Lemma 3.9. Let $\mathcal{C}$ be a complete family, let $G$ be a graph, and let $H$ be a connected subgraph of $G$ in $\mathcal{C}$. Let $E''$ be a minimal subset of $E(G)$ such that every component of $G - E''$ is in $\mathcal{C}$; let $E^{**}$ be a minimal subset of $E(G/H)$ such that every component of $(G/H) - E^{**}$ is in $\mathcal{C}$; and let

$$E' = \{e \in E(G) \mid e \text{ is in no subgraph of } G \text{ in } \mathcal{C}\}$$

and

$$E^* = \{e \in E(G/H) \mid e \text{ is in no subgraph of } G/H \text{ in } \mathcal{C}\}.$$

Then

$$E'' = E' = E^* = E^{**}. \quad (23)$$

Proof. The first and last equalities of (23) are instances of Corollary 3.8. It remains to prove $E' = E^*$.

Let $H$ be a connected subgraph of $G$ where $H \in \mathcal{C}$, let $e \in E'$, and suppose $e \notin E^*$, by way of contradiction. Then $e$ is in a subgraph $H''$ of $G/H$ where $H'' \in \mathcal{C}$. Denote by $G''$ the subgraph of $G$ induced by $E(H) \cup E(H'')$. Thus,

$$H \subseteq G'', \quad H \in \mathcal{C}, \quad G''/H = H'' \in \mathcal{C},$$

and so by (C3), $G'' \in \mathcal{C}$. But, $e \in E(H'') \subseteq E(G'')$, contrary to $e \in E'$. Therefore,

$$E' \subseteq E^*. \quad (24)$$

Let $e \in E(G) - E'$. Hence by Corollary 3.8, $G$ has a unique maximal subgraph $H_0 \in \mathcal{C}$ such that $e \in E(H_0)$. If $H$ and $H_0$ are disjoint, then $e \in E(H_0), H_0 \subseteq G/H$, and $H_0 \in \mathcal{C}$ jointly imply

$$e \notin E^*. \quad (25)$$

Since (25) holds whenever $e \notin E'$, (24) implies $E' = E^*$. \qed

Let $\mathcal{C} = \{C_3\}$ (not a complete family) and let $G$ be the graph with $V(G) = \{a, b, c, d, e\}$ and

$$E(G) = \{ab, bc, cd, de, ea, ac, ce\}.$$

Now consider what happens if subgraphs in $\mathcal{C}$ (i.e., 3-cycles) are contracted until none remain. If $H = G[\{a, c, e\}]$ is contracted, then $G/H$ has order 3 and no subgraph in
4. Free families and reduced graphs

Let \( \mathcal{C} \) be a complete family and let \( G \) be a graph. By Corollary 3.8, \( G \) has a unique maximal spanning subgraph

\[
G' = G - E'' = G - E'
\]

(where \( E'' \) and \( E' \) are the sets of Corollary 3.8), with components in \( \mathcal{C} \). Denote the components of \( G' \) by \( \{H_1, H_2, \ldots, H_c\} \). Define the \( \mathcal{C} \)-reduction of \( G \), called \( G/\mathcal{C} \), to be the graph obtained from \( G \) by contracting each \( H_i \) (1 \( \leq i \leq c \)) to a distinct vertex and by removing any resulting loops. If \( G \) has no nontrivial subgraph in \( \mathcal{C} \), then \( G = G/\mathcal{C} \), and we call \( G \) \( \mathcal{C} \)-reduced. For any family \( \mathcal{F} \), and for any graph \( G \), the \( \mathcal{F}^o \)-reduction of \( G \) is \( K_1 \) if and only if \( G \) is in the kernel \( \mathcal{F}^o \) of \( \mathcal{F} \).

**Theorem 4.1.** If \( \mathcal{C} \) is a complete family and \( G \) is a graph, then the \( \mathcal{C} \)-reduction of \( G \), i.e., \( G/\mathcal{C} \), is the unique \( \mathcal{C} \)-reduced graph obtained from \( G \) by contractions of subgraphs in \( \mathcal{C} \).

**Proof.** Let \( \mathcal{C} \) be a complete family, let \( G \) be a graph, and let \( E'' \) and \( E' \) have the meaning of Lemma 3.9 (and of Corollary 3.8). Let \( G_1 \) be a reduced graph obtained from \( G \) by a sequence of contractions of connected subgraphs of \( G \) in \( \mathcal{C} \). As \( G \) is contracted to \( G_1 \) by a sequence of contractions of connected subgraphs of \( G \), Lemma 3.9 asserts that \( E'' \) and \( E' \) remain constant and equal throughout every step of the sequence. Since \( G_1 \) is \( \mathcal{C} \)-reduced, \( G_1 \) has no edge in any subgraph in \( \mathcal{C} \), and so \( E(G_1) \subseteq E' \). As \( G \) is contracted to \( G_1 \), the only edges that are contracted are edges in subgraphs in \( \mathcal{C} \), and so the constancy of \( E' \) implies \( E' \subseteq E(G_1) \). Hence, \( E(G_1) = E' = E'' \) and by definition, \( G_1 \) must be \( G/\mathcal{C} \). \( \square \)

For any complete family \( \mathcal{C} \), the family \( \mathcal{C}^R \) (defined in (4)) is the family of \( \mathcal{C} \)-reduced graphs.

**Corollary 4.2.** Let \( \mathcal{C}' \) and \( \mathcal{C}'' \) be complete families of graphs. If \( \mathcal{C}' \subseteq \mathcal{C}'' \) then \( (\mathcal{C}'')^R \subseteq (\mathcal{C}')^R \).

**Proof.** If \( G \in (\mathcal{C}'')^R \), then \( G \) is \( \mathcal{C}'' \)-reduced, and so \( G = G/\mathcal{C}'' \). By Theorem 4.1, \( G/\mathcal{C}'' \) has no nontrivial subgraph in \( \mathcal{C}'' \). Since \( \mathcal{C}' \subseteq \mathcal{C}'' \), \( G/\mathcal{C}' \) thus has no nontrivial subgraph in \( \mathcal{C}' \), and hence by definition, \( G/\mathcal{C}' \) is \( \mathcal{C}' \)-reduced. Hence \( G \in (\mathcal{C}')^R \). \( \square \)
There is a duality between complete families and free families, and between the operations $\mathcal{C} \rightarrow \mathcal{C}^R$ and $\mathcal{F} \rightarrow \mathcal{F}^C$, where $\mathcal{C}$ is complete and $\mathcal{F}$ is free. This duality appears below, and it has been studied further in [5]. For our purposes here, a contraction is trivial whenever it is edgeless, and any graph with an edge is a nontrivial contraction of itself.

**Lemma 4.3.** For any family $\mathcal{C}$, if $H$ is a subgraph of $G$ and if $G \in \mathcal{C}^R$, then $H \in \mathcal{C}^R$.

**Proof.** By the definition of $\mathcal{C}^R$, since $G \in \mathcal{C}^R$, $G$ is $\mathcal{C}$-reduced. By definition, any subgraph $H$ of $G$ is $\mathcal{C}$-reduced, and hence $H \in \mathcal{C}^R$. □

**Lemma 4.4.** For any family $\mathcal{C}$, any graph in $\mathcal{C} \cap \mathcal{C}^R$ is edgeless.

**Proof.** If $H \in \mathcal{C}^R$, then by definition $H$ has no nontrivial subgraph in $\mathcal{C}$. □

**Lemma 4.5.** For any family $\mathcal{F}$, any graph in $\mathcal{F} \cap \mathcal{F}^C$ is edgeless.

**Proof.** If $G \in \mathcal{F}^C$ then no nontrivial contraction of $G$ is in $\mathcal{F}$. □

**Theorem 4.6.** For any family $\mathcal{C}$ that is closed under contraction, $\mathcal{C}^R$ is a free family.

**Proof.** We show that $\mathcal{C}^R$ satisfies (F1)–(F3). By definition, all edgeless graphs are in $\mathcal{C}^R$, so (F1) holds. By Lemma 4.3, $\mathcal{C}^R$ satisfies (F2).

Suppose by contradiction that (F3) fails for $G$ and some nontrivial induced subgraph $H$ of $G$. Thus, $H \in \mathcal{C}^R$, $G/H \in \mathcal{C}^R$, but $G \notin \mathcal{C}^R$, and hence $G$ has a nontrivial subgraph $G' \in \mathcal{C}$.

First, suppose $V(G') \subseteq V(H)$. Since $H$ is an induced subgraph, $G' \subseteq H$. Since $H \in \mathcal{C}^R$, Lemma 4.3 implies that $G' \in \mathcal{C}^R$, too. Thus, $G' \in \mathcal{C} \cap \mathcal{C}^R$, which is impossible by Lemma 4.4.

Therefore, $V(G') \not\subseteq V(H)$, and so $G'/(H \cap G')$ is nontrivial, where $G'/(H \cap G')$ denotes $G'$ is $H \cap G'$ is edgeless. Since $\mathcal{C}$ is closed under contraction and $G' \in \mathcal{C}$, we have $G'/(H' \cap G) \in \mathcal{C}$. Thus, $G/H$ has the nontrivial subgraph $G'/(H \cap G')$ in $\mathcal{C}$, contrary to $G/H \in \mathcal{C}^R$. Hence, (F3) holds for $\mathcal{C}^R$, and so $\mathcal{C}^R$ is free. □

Closure under contraction is needed in Theorem 4.6. Let $\mathcal{C}$ be the family of all graphs of odd order. Then $\mathcal{C}$ is not closed under contraction. Clearly, $K_2 \in \mathcal{C}^R$. Suppose that $\mathcal{C}^R$ is free. Then (F3) and $K_2 \in \mathcal{C}^R$ imply that $\mathcal{C}^R$ contains trees of all odd orders. So does $\mathcal{C}$. This violates Lemma 4.4.

**Lemma 4.7.** Let $\mathcal{F}$ be a free family containing $K_2$ as a member. The subfamily of connected graphs in $\mathcal{F}^C$ is closed under edge-addition.

**Proof.** Let $\mathcal{F}$ be a free family containing $K_2$ as a member, and let $G$ be a nontrivial graph with a distinguished edge $e$ such that $H = G - e$ is connected. By contradiction,
suppose that \( H \in \mathcal{F}^C \) and \( G \notin \mathcal{F}^C \). Then \( G \) has a nontrivial contraction \( G_0 \) (say) in \( \mathcal{F} \), but \( H \) has no nontrivial contraction in \( \mathcal{F} \).

Case 1: Suppose \( e \notin E(G_0) \). Let \( G_0(e) \) denote the graph to which \( G \) is contracted when the edges of \((E(G) − E(G_0)) − e\) are contracted. First suppose that \( e \notin E(G_0(e)) \). Then the contraction (in \( G \)) of the edges of \((E(G) − E(G_0)) − e\) identifies the ends of \( e \), and hence \( G_0 = G_0(e) \) and this \( G_0(e) \) is also a contraction of \( H = G − e \). But then \( H \) has a nontrivial contraction \( G_0 \) in \( \mathcal{F} \), a contradiction. Therefore, \( e \in E(G_0(e)) \), and \( G_0 \) is obtained from \( G_0(e) \) by contracting \( e \). If \( G_0(e) \) has an edge \( e' \) parallel to \( e \), then \( G_0 \in \mathcal{F} \) could be obtained from \( H \) by contracting \( H \) to \( G_0(e) − e \) and then by contracting \( e' \), but this would violate the fact that \( H \) has no nontrivial contraction in \( \mathcal{F} \). Hence, \( G_0(e) \) has no edge \( e' \) parallel to \( e \), and so \( G_0(e)[e] \), a \( K_2 \), is an induced subgraph of \( G_0(e) \).

Since \( \mathcal{F} \) is a free family, since \( G_0(e)[e] = K_2 \in \mathcal{F} \), and since \( G_0(e)/e = G_0 \in \mathcal{F} \), (F3) implies that \( G_0(e) \in \mathcal{F} \). By (F2), \( G_0(e) − e \in \mathcal{F} \). Since \( G − e \) is connected, so is \( G_0(e) − e \), and it is nontrivial. Hence, \( H = G − e \) has the nontrivial contraction \( G_0(e) − e \in \mathcal{F} \), a contradiction precluding Case 1.

Case 2: Suppose \( e \in E(G_0) \). By \( G_0 \in \mathcal{F} \) and by (F2), \( G_0 − e \in \mathcal{F} \). Since \( G − e \) is connected, so is \( G_0 − e \), and so \( G_0 − e \) is a nontrivial contraction of \( H \) lying in \( \mathcal{F} \), contrary to \( H \in \mathcal{F}^C \). □

Lemma 4.8. For any family \( \mathcal{F} \), \( \mathcal{F}^C \) is closed under contraction.

Proof. Let \( \mathcal{F} \) be a family. If all members of \( \mathcal{F}^C \) are edgeless, then the lemma is easy.

Suppose that \( G \in \mathcal{F}^C \) and that \( G_0 \) is a nontrivial contraction of \( G \). By the definition of \( \mathcal{F}^C \), \( G \) has no nontrivial contraction in \( \mathcal{F} \), and so neither does \( G_0 \). Thus, \( G_0 \in \mathcal{F}^C \). □

Lemma 4.9. If \( \mathcal{F} \) is free and \( G \in \mathcal{F} \), then \( G \cup K_1 \in \mathcal{F} \).

Proof. Apply (F3) with \( H \) and \( G \), respectively, of (F3) replaced by \( G \) and \( G \cup K_1 \), respectively. Then \( G/H \) of (F3) is edgeless, and by (F1) it is in \( \mathcal{F} \). □

Theorem 4.10. Suppose \( \mathcal{F} \) is a free family. Then the family \( \mathcal{C} = \mathcal{F}^C \) is complete. Also, \( \mathcal{F} = \mathcal{C}^R = (\mathcal{F}^C)^R \).

Proof. If no graph in \( \mathcal{F} \) has an edge, then \( \mathcal{F} \) is the family of all edgeless graphs, \( \mathcal{C} = \mathcal{F}^C \) is the family of all graphs, which is complete, and \( \mathcal{C}^R \) is the family of all edgeless graphs.

Suppose that \( \mathcal{F} \) is a free family such that some graph of \( \mathcal{F} \) has an edge, and let \( \mathcal{C} = \mathcal{F}^C \). By (F2), \( K_2 \in \mathcal{F} \), so Lemma 4.7 applies. We must prove that \( \mathcal{C} \) satisfies axioms (C1)–(C3) of the definition of a complete family, and that \( \mathcal{F} = \mathcal{C}^R \). By definition, \( \mathcal{C} \) satisfies (C1). By Lemma 4.8, (C2) holds.
We prove (C3). Let $G$ be a supergraph of a nontrivial graph

\[ H \in \mathcal{C}. \]  \hspace{1cm} (26)

We claim

\[ G/H \in \mathcal{C} \Rightarrow G \in \mathcal{C}. \]  \hspace{1cm} (27)

By way of contradiction, suppose (27) is false. Then

\[ G \notin \mathcal{C} \quad \text{and} \quad G/H \in \mathcal{C}. \]  \hspace{1cm} (28)

By the definition of $\mathcal{C}$, $G \notin \mathcal{C}$ of (28) implies that $G$ has a nontrivial contraction $G_0$ (say) in $\mathcal{F}$. Let $\theta : V(G) \rightarrow V(G_0)$ denote the surjection induced by this contraction.

We claim first that there is an edge $e \in E(H) \cap E(G_0)$: otherwise, $G/H$ can be contracted to the nontrivial graph $G_0 \in \mathcal{F}$, contrary to $G/H \in \mathcal{C} = \mathcal{F}^C$ in (28). Let $H_e$ be the component of $H$ containing $e$. Denote

\[ E = \{xy \mid \text{there is an } i \text{ such that } x, y \in \theta^{-1}(v_i) \cap H_e\}. \]

Let $J = (H/E)(H - E(H_e))$. Note $J \in \mathcal{F}^C$. Let $H_0$ be the subgraph of $G_0$ containing the edges of $H_e \cap G_0$ and no isolated vertices. Note that $H_0 \notin \mathcal{F}$. Add enough isolated vertices to $H_0$ so that it will equal $J$. By Lemma 4.9, $J \in \mathcal{F}$, contradicting Lemma 4.5. This contradiction proves (27) and hence that $\mathcal{C}$ satisfies (C3).

Now we prove $\mathcal{F} \subseteq \mathcal{C}^R$. Suppose $G \in \mathcal{F}$. By contradiction, if $G \notin \mathcal{C}^R$ then $G$ has a nontrivial subgraph $H \in \mathcal{C} = \mathcal{C}^C$. By $G \in \mathcal{F}$ and (F2), $H \in \mathcal{F}$, and so by Lemma 4.5, $H$ is trivial, a contradiction.

To prove $\mathcal{C}^R \subseteq \mathcal{F}$, we suppose (by contradiction) that $G$ is a minimal member of $\mathcal{C}^R - \mathcal{F}$. Since $\mathcal{F}$ contains all edgeless graphs, $G$ is a nontrivial graph in $\mathcal{C}^R$. By Lemma 4.4, $G \notin \mathcal{C} = \mathcal{C}^C$. One of these two cases holds:

**Case A:** Suppose $G$ is disconnected. Let $H$ be a component of $G$ and let $H' = G - H$.

By the minimality of $G$, both $H$ and $H'$ are in $\mathcal{F}$. Let $G'$ denote the graph obtained by adding an edge $e$ (say) joining some vertex of $V(H)$ and some vertex $V(H')$. Therefore, $G'$ has vertex-induced subgraphs $G'[e], H,$ and $H'$, all in $\mathcal{F}$ since $K_2 \in \mathcal{F}$.

By two applications of (F3), $G' \in \mathcal{F}$. By (F2), $G = G' - e \in \mathcal{F}$, a contradiction.

**Case B:** Suppose $G$ is connected. Since $G \notin \mathcal{F}^C$, some nontrivial contraction $G_0$ (say) of $G$ is in $\mathcal{F}$. Since $G \notin \mathcal{F}$, $G \neq G_0$. Since $G$ is connected and $G_0 \neq K_1$, we have $E(G_0) \neq \emptyset$. Hence, $G - E(G_0)$ has $|V(G_0)| = c$ components, say $H_1, H_2, \ldots, H_c$, for some $c \geq 2$. Each $H_i$ is an induced subgraph of $G$, and by Lemma 4.3, $H_i \in \mathcal{C}^R$ ($1 \leq i \leq c$). Since $G$ was chosen to be a minimal member of $\mathcal{C}^R - \mathcal{F}$ and since $c \geq 2$, each $H_i$ ($1 \leq i \leq c$) is in $\mathcal{F}$. But also $G_0 \in \mathcal{F}$, and so by repeated applications of axiom (F3), $G \in \mathcal{F}$. This contradiction proves $\mathcal{C}^R = \mathcal{F}$, as claimed. ∎

In Theorem 4.10, $\mathcal{F}$ cannot be just any family. Suppose, for example, that $\mathcal{F}$ is the family of connected graphs of odd order. Thus, $\mathcal{F}$ violates (F2), so $\mathcal{F}$ is not a free family. It is easily seen that $\mathcal{F}^C$ is not complete: $\mathcal{F}^C$ contains $K_2$, and hence
if (C3) held then \( \mathcal{F}^C \) would contain all trees. But trees of odd order are in \( \mathcal{F} \), and Lemma 4.5 is violated.

**Theorem 4.11.** If \( \mathcal{C} \) is a complete family, then \((\mathcal{C}^R)^C = \mathcal{C}\).

**Proof.** Suppose that \( \mathcal{C} \) is complete and let \( \mathcal{F} = \mathcal{C}^R \). First suppose \( G \in (\mathcal{C}^R)^C \). By the definition of \( \mathcal{F}^C \), no nontrivial contraction \( H \) of \( G \) is in \( \mathcal{C}^R \). But by Theorem 4.1, the graph \( G/H \) is a contraction of \( G \) in \( \mathcal{C}^R \). Hence, \( G/H \) must be edgeless, and this implies that the components of \( G \) are in \( \mathcal{C} \). Hence by Theorem 3.7, \( G \in \mathcal{C} \), and so \((\mathcal{C}^R)^C \subseteq \mathcal{C}\).

Suppose instead that \( G \in \mathcal{C} \). The complete family \( \mathcal{C} \) is closed under contraction and hence all contractions of \( G \) are in \( \mathcal{C} \). Thus, by Lemma 4.4, \( G \) has no nontrivial contraction in \( \mathcal{C}^R \), and so by the definition of \( \mathcal{F}^C \), \( G \in (\mathcal{C}^R)^C \). Thus, \( \mathcal{C} \subseteq (\mathcal{C}^R)^C \). □

**Theorem 4.12.** Let \( \mathcal{C} \) and \( \mathcal{F} \) be two graph families. If both \( \mathcal{C} = \mathcal{F}^C \) and \( \mathcal{F} = \mathcal{C}^R \), then \( \mathcal{C} \) is a complete family and \( \mathcal{F} \) is a free family. For any complete family \( \mathcal{C} \) there is a free family \( \mathcal{F} = \mathcal{F}^C \) such that \( \mathcal{C} = \mathcal{F}^C \). For any free family \( \mathcal{F} \) there is a complete family \( \mathcal{C} = \mathcal{F}^C \) such that \( \mathcal{F} = \mathcal{C}^R \).

**Proof.** Let \( \mathcal{C} \) and \( \mathcal{F} \) be two graph families, and suppose \( \mathcal{C} = \mathcal{F}^C \) and \( \mathcal{F} = \mathcal{C}^R \). By Lemma 4.8, \( \mathcal{C} = \mathcal{F}^C \) is closed under contraction. Hence, by Theorem 4.6, \( \mathcal{F} = \mathcal{C}^R \) is a free family, and so by Theorem 4.10, \( \mathcal{C} = \mathcal{F}^C \) is a complete family.

For any complete family \( \mathcal{C} \), apply Theorems 4.6 and 4.11 to obtain the desired free family \( \mathcal{F} = \mathcal{C}^R \). For any free family \( \mathcal{F} \), apply Theorem 4.10 to obtain the desired complete family \( \mathcal{C} = \mathcal{F}^C \). □

For the operations \( \mathcal{C} \rightarrow \mathcal{C}^R \) and \( \mathcal{F} \rightarrow \mathcal{F}^C \), it is natural to ask when families \( \mathcal{C} \) and \( \mathcal{F} \) exist satisfying \( \mathcal{C} = \mathcal{F}^C \) and \( \mathcal{F} = \mathcal{C}^R \). Thus, Theorem 4.12 motivates the study of complete families and free families. Our original motivation for considering these families was the study of the kernel \( \mathcal{F}^O \) and the corresponding reduced graphs, but Theorem 4.12 is another justification.

**Theorem 4.13.** Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be free families of graphs. Then

\[ \mathcal{F}_1 \subseteq \mathcal{F}_2 \quad \text{if and only if} \quad \mathcal{F}_2^C \subseteq \mathcal{F}_1^C. \]

**Proof.** Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be free families. Suppose \( \mathcal{F}_1 \subseteq \mathcal{F}_2 \) and let \( G \in \mathcal{F}_2^C \). By definition, no nontrivial contraction of \( G \) is in \( \mathcal{F}_2 \). Hence, no nontrivial contraction of \( G \) is in \( \mathcal{F}_1 \), and so by definition, \( G \in \mathcal{F}_1^C \).

Conversely, suppose \( \mathcal{F}_2^C \subseteq \mathcal{F}_1^C \). By Theorem 4.10, \( \mathcal{F}_2^C \) and \( \mathcal{F}_1^C \) are complete families. By Theorem 4.10 (twice) and Corollary 4.2,

\[ \mathcal{F}_1 = (\mathcal{F}_1^C)^R \subseteq (\mathcal{F}_2^C)^R = \mathcal{F}_2. \]
Corollary 4.14. Let $\mathcal{C}'$ and $\mathcal{C}''$ be complete families. Then

$$\mathcal{C}' \subseteq \mathcal{C}'' \text{ if and only if } \mathcal{C}''^R \subseteq \mathcal{C}'^R.$$ 

Proof. By Theorem 4.6, $\mathcal{F}_1 = (\mathcal{C}'')^R$ and $\mathcal{F}_2 = (\mathcal{C}')^R$ are free families. This and Theorem 4.11 imply both $\mathcal{F}_1^C = (\mathcal{C}'')^R)^C = \mathcal{C}''$ and $\mathcal{F}_2^C = ((\mathcal{C}')^R)^C = \mathcal{C}'$. Applying Theorem 4.13, we get the result. $\Box$

5. Examples: free families

The smallest free family $\mathcal{F}$ containing a nontrivial graph is the family of all forests. (By (F2), if a free family $\mathcal{F}$ has any member with an edge, then $K_2 \in \mathcal{F}$. This and (F1) and (F3) imply that $\mathcal{F}$ contains all forests.) The corresponding complete family $\mathcal{F}^C$ consists of all graphs with no cut-edges.

Corresponding to edge-connectivity $\kappa'(G)$, define

$$\overline{\kappa}'(G) = \max_{H \subseteq G} \kappa'(H).$$

Let $k \in \mathbb{N}$. If $\mathcal{C}$ is the complete family of graphs with $k$-edge-connected components, then $\mathcal{C}^R = \{ G \mid \overline{\kappa}'(G) < k \}$ is the corresponding free family.

For $k \geq 2$, define $\mathcal{F}_k = \{ G \mid G \text{ has girth at least } k \}$. Then $\mathcal{F}_k$ is a free family, $\mathcal{F}_2$ is the family of all graphs, and $\mathcal{F}_3$ is the family of all simple graphs.

Define, for any nontrivial graph $G$,

$$\gamma(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)| - 1},$$

where the maximum runs over all nontrivial subgraphs $H$ of $G$. Nash-Williams [6] showed that $[\gamma(G)]$, called the edge-arboricity of $G$, is the minimum number of forests whose union contains $G$. For $k \in \mathbb{N}$, the family of graphs with edge-arboricity at most $k$ is a free family. If $\mathcal{C}$ is the complete family of graphs with $k$ edge-disjoint spanning trees, then $\mathcal{C}^R$ is the family of graphs $G$ with edge-arboricity at most $k$, but with no nontrivial subgraph of $G$ having $k$ edge-disjoint spanning trees.

Suppose a free family $\mathcal{F}$ contains a graph having an $n$-cycle. By (F2), $K_2, C_n \in \mathcal{F}$. This and repeated applications of (F3) imply that all cycles of length at least $n$ are in $\mathcal{F}$. For example, the free families $\mathcal{C}L^R$ and $(\mathcal{C}L^O)^R$ contain all cycles of length at least 4.

The complete family of graphs whose components all have two edge-disjoint spanning trees is contained (by Theorem 2 and the corollary of Theorem 3 of [2]) in the kernel $\mathcal{L}^O$, a complete family, by Theorem 3.3. Hence, by Corollary 4.14, any graph $G$ in $(\mathcal{L}^O)^R$ has edge-arboricity at most 2.
References