Compute the Derivative by Definition: The four step procedure

Given a function \( f(x) \), the definition of the derivative of \( f(x) \), is

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \quad \text{and at } x = a, \ f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},
\]
provided these limits exist. Given a function \( f(x) \), we can follow the following steps to find \( f'(x) \).

**Step 1** Evaluate \( f(x+h) \) and \( f(x) \).

**Step 2** Compute \( f(x+h) - f(x) \). Combine like terms. If \( h \) is a common factor of the terms, factor the expression by removing the common factor \( h \).

**Step 3** Simply \( \frac{f(x+h) - f(x)}{h} \). As \( h \to 0 \) in the last step, we **must** cancel the zero factor \( h \) in the denominator in Step 3.

**Step 4** Compute \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \) by letting \( h \to 0 \) in the simplified expression.

**Example 1** Let \( f(x) = ax^2 + bx + c \). Compute \( f'(x) \) by the definition (that is, use the four step process).

**Solution:** Step 1, write

\[
f(x) = (ax^2 + bx + c).
\]

Step 2: Use algebra to single out the factor \( h \).

\[
f(x+h) - f(x) = (a(x+h)^2 + b(x+h) + c) - (ax^2 + bx + c) = 2axh + ah^2 + bx + bh + c.
\]

Step 3: Cancel the zero factor \( h \) is the most important thing in this step.

\[
\frac{f(x+h) - f(x)}{h} = 2ax + ah + b.
\]

Step 4: Let \( h \to 0 \) in the resulted expression in Step 3.

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 2ax + ah + b = 2ax + 0 + b = 2a + b.
\]

**Example 2** Let \( f(x) = \frac{1}{x+1} \). Compute \( f'(x) \) by the definition (that is, use the four step process).

**Solution:** Step 1, write

\[
f(x+h) = \frac{1}{(x+h)+1} = \frac{1}{x+h+1}.
\]

Step 2: Use algebra to single out the factor \( h \).

\[
f(x+h) - f(x) = \frac{1}{x+h+1} - \frac{1}{x+1} = \frac{(x+1) - (x+h+1)}{(x+h+1)(x+1)} = \frac{h}{(x+h+1)(x+1)}.
\]

Step 3: Cancel the zero factor \( h \) is the most important in this step.

\[
\frac{f(x+h) - f(x)}{h} = \frac{1}{h} \left[ \frac{1}{x+h+1} - \frac{1}{x+1} \right] = \frac{1}{h} \left[ \frac{-h}{(x+h+1)(x+1)} \right] = \frac{-1}{(x+h+1)(x+1)}.
\]

Step 4: Let \( h \to 0 \) in the resulted expression in Step 3.

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-1}{(x+h+1)(x+1)} = \frac{-1}{(x+0+1)(x+1)} = \frac{-1}{(x+1)^2}.
\]
Example 3 Let \( f(x) = \sqrt{2x+5} \). Compute \( f'(x) \) by the definition (that is, use the four step process).

Solution: Step 1, write
\[
f(x + h) = \sqrt{2(x + h) + 5} = \sqrt{2x + 2h + 5}.
\]
Step 2: Use algebra to single out the factor \( h \). Here we need the identity \((A + B)(A - B) = A^2 - B^2\) to get rid of the square root so that \( h \) can be factored out.
\[
f(x + h) - f(x) = \sqrt{2x + 2h + 5} - \sqrt{2x + 5} = \frac{(2x + 2h + 5) - (2x + 5)}{\sqrt{2x + 2h + 5} + \sqrt{2x + 5}} = \frac{2h}{\sqrt{2x + 2h + 5} + \sqrt{2x + 5}}.
\]
Step 3: Cancel the zero factor \( h \) is the most important thing in this step.
\[
f(x + h) - f(x) = \frac{2h}{h} \left[ \frac{1}{\sqrt{2x + 2h + 5} + \sqrt{2x + 5}} \right] = \frac{2}{\sqrt{2x + 2h + 5} + \sqrt{2x + 5}}.
\]
Step 4: Let \( h \to 0 \) in the resulted expression in Step 3.
\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{2}{h} \left[ \frac{1}{2x + 2h + 5 + 2x + 5} \right] = \frac{2}{2x + 0 + 0 + 2x + 5} = \frac{1}{\sqrt{2x + 5}}.
\]

Find Derivatives by Using Differentiation Rules

Some Differentiation Rules and Higher Order of Derivatives:

1. Derivative of a constant: Let \( C \) be a constant, then \( \frac{d}{dx}C = 0 \).
2. Power Rule: For a real number \( n \),
\[
\frac{dx^n}{dx} = nx^{n-1}.
\]
3. Linear Property: For a constant \( c \) and functions \( f(x) \) and \( g(x) \),
\[
[f(x) \pm g(x)]' = f'(x) \pm g'(x) \quad \text{and} \quad [cf(x)]' = cf'(x).
\]
4. Higher Order of Derivative: \( f^{(n)}(x) = \frac{d}{dx} f'(x), f''(x) = \frac{d}{dx} f''(x), f^{(4)}(x) = \frac{d}{dx} f^{(4)}(x), f^{(5)}(x) = \frac{d}{dx} f^{(5)}(x), \ldots \).

Example 1 Find the derivative of \( f(x) = \frac{3}{x} - 1 + \sqrt{x} + 5x^4 \).

Solution: Write every term in the power form so that the power rule can be easily applied:
\[
f(x) = \frac{3}{x} - 1 + \sqrt{x} + 5x^4 = 3x^{-1} - 1 + x^{\frac{1}{2}} + 5x^4.
\]
Apply the power rules to get \( f'(x) = 3 \cdot (-1)x^{-1} - 0 + \frac{1}{2}x^{-\frac{1}{2}} + 5 \cdot 4x^{4-1} = -\frac{1}{x^2} + \frac{1}{2}x^{-\frac{1}{2}} + 20x^3 \).

Example 2 Given \( f(x) = \frac{x^2 - x + 1}{\sqrt{x}} \), find \( f''(x) \).

Solution: Write every term in the power form so that the power rule can be easily applied:
\[
f(x) = \frac{x^2 - x + 1}{\sqrt{x}} = \frac{x^2}{\sqrt{x}} - \frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}} = x^{\frac{3}{2}} - x^{\frac{1}{2}} + x^{-\frac{1}{2}}.
\]
Apply the rules to get
\[
f'(x) = \frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}, \quad f''(x) = \frac{3}{4}x^{-\frac{1}{2}} + \frac{1}{4}x^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{5}{2}}, \quad f'''(x) = \frac{3}{8}x^{-\frac{3}{2}} - \frac{3}{8}x^{-\frac{5}{2}} - \frac{15}{8}x^{-\frac{7}{2}}.
\]