Application of Derivatives to Optimization Problems

Facts: Let \( f(x) \) be a differentiable function on an interval \( I \).

1. (Where to seek optimal solutions?) Any local extrema must occur at a critical point.
2. (How does \( f'(x) \) play a role?) The sign of \( f'(x) \) can predict whether \( f(x) \) is increasing or decreasing.

Strategy to find optimal solutions: Firstly we determine the critical points, and the sign of \( f'(x) \) in the subintervals determined by the critical points. Then interpret the sign of \( f'(x) \) to discuss whether an optimal solution occurs at a given critical point.

Example 1 A three side fence is to be built next to a straight section of river, which forms the forth side of a rectangular region. The enclosed area is to equal \( 1800 \text{ ft}^2 \). Find the minimum perimeter and the dimensions of the corresponding enclosure.

Solution: Let the length of the two sides of the rectangular region be \( x \) and \( y \), where \( y \) represents the length of the side parallel to the river bank. Then as \( xy = 1800 \), \( y = \frac{1800}{x} \). The perimeter of the region is a function of \( x \):

\[
f(x) = 2x + y = 2x + \frac{1800}{x}.
\]

As \( x \) is a length, \( x > 0 \). Therefore, the domain of \( f(x) \) is \((0, \infty)\).

Compute \( f'(x) = 2 - \frac{1800}{x^2} \). Set \( f'(x) = 0 \) to get \( x = \pm 30 \). Only \( x = 30 \) is in the domain. Moreover, \( x = 30 \) partitions the domain \((0, \infty)\) into two subintervals: \((0, 30)\) and \((30, \infty)\). As \( f'(1) < 0 \) and \( f'(40) > 0 \), \( f(x) \) is decreasing in \((0, 30)\) and increasing in \((30, \infty)\). The shape of graph of \( y = f(x) \) suggests that \( f(x) \) has an absolute minimum at \( x = 30 \). Thus the minimum perimeter and the corresponding dimensions are, respectively,

\[
f(30) = 2 \cdot 30 + \frac{1800}{30} = 120, \quad x = 30, \quad y = \frac{1800}{30} = 60.
\]

Example 2 Find the point on the curve \( y = x^2 \) closest to the point \( Q = (0, 1) \).

Solution: Let \( P \) be a point of the curve \( y = x^2 \). Then \( P = (x, x^2) \). The distance between \( P \) and \( Q \) is a function of \( x \):

\[
f(x) = \sqrt{(x-0)^2 + (x^2-1)^2} = \sqrt{x^2 + x^2 - 2x^2 + 1} = \sqrt{x^4 - x^2 + 1} = (x^4 - x^2 + 1)^{\frac{1}{2}}.
\]
As \( x^4 - x^2 + 1 > 0 \) for all \( x \), the domain of \( f(x) \) is \((−∞, ∞)\). Compute \( f'(x) \) by using the chain rule

\[
f'(x) = \frac{1}{2} \frac{4x^3 - 2x}{\sqrt{x^4 - x^2 + 1}} = \frac{x(2x^2 - 1)}{\sqrt{x^4 - x^2 + 1}}.
\]

Set \( f'(x) = 0 \). Then the numerator must equal zero. Thus the critical numbers are

\[
x = 0, x = \frac{1}{\sqrt{2}}, x = -\frac{1}{\sqrt{2}}.
\]

These points partition the domain \((−∞, ∞)\) into four intervals:

\[
(−∞, −\frac{1}{\sqrt{2}}), (−\frac{1}{\sqrt{2}}, 0), (0, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, ∞).
\]

Compute \( f'(-10) < 0 \), \( f'(-\frac{1}{2}) > 0 \), \( f'\left(\frac{1}{2}\right) < 0 \) and \( f'(10) > 0 \), (note that the denominator is always positive). Therefore, \( f(x) \) is increasing in \((−\frac{1}{\sqrt{2}}, 0) \cup \left(\frac{1}{\sqrt{2}}, ∞\right)\), and decreasing in \((−∞, −\frac{1}{\sqrt{2}}) \cup (0, \frac{1}{\sqrt{2}})\).

From the shape of the graph \( y = f(x) \), \( f(x) \) has local minimum at \( x = ±\frac{1}{\sqrt{2}} \), and at least one of them gives the absolute minimum. Compute

\[
f\left(\frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{4}}.
\]

Therefore, there are two points \((±\frac{1}{\sqrt{2}}, \frac{1}{2})\) which are on \( y = x^2 \) and are closest to \((0, 1)\).

**Example 3** Determine two real numbers with difference 20 and minimum possible product.

**Solution:** Let \( x \) and \( y \) denote these two real numbers with \( x \leq y \). Then \( y = 20 + x \). Thus we want to minimize

\[
f(x) = xy = x(20 + x) = 20x + x^2;
\]

in \((−∞, ∞)\). Note that \( f'(x) = 2x + 20 \), and so \( x = −10 \) is the only critical number. As \( f'(x) < 0 \) when \( x < −10 \) and \( f'(x) > 0 \) when \( x > −10 \). Therefore \( f(−10) \) is the only local minimum value of \( f(x) \) in \((−∞, ∞)\), and so it is also the absolute minimum value of \( f(x) \) in its domain. Hence \( x = −10 \) and \( y = 10 \) are the two numbers we want.