The Mean Value Theorems

Rolle’s Theorem Suppose that \( f(x) \) is continuous on \([a, b]\) and is differentiable in \((a, b)\). If \( f(a) = f(b) \), then there exists a point \( c \) in \((a, b)\) such that \( f'(c) = 0 \).

The Mean Value Theorem Suppose that \( f(x) \) is continuous on \([a, b]\) and is differentiable in \((a, b)\). Then there exists a point \( c \) in \((a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

Example 1 Given \( f(x) = \frac{1}{x} \) on the interval \([-1, 1]\), show that there is no value of \( c \) in the interval \([-1, 1]\) that makes the conclusion of the Mean Value Theorem true.

Solution: Note that \( f'(x) = -\frac{1}{x^2} \). Here \( a = -1 \) and \( b = 1 \). The Mean Value Theorem would assure that there is a number \( c \) such that

\[
-\frac{1}{c^2} = f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{2}{2} = 1.
\]

Since a negative number cannot be equal to a positive number, we cannot find such a number \( c \).

Why this would happen? This is because that \( f(x) \) has a discontinuity \( x = 0 \) inside the interval \([-1, 1]\) and so the hypothesis of the Mean Value Theorem is not satisfied.

Example 2 Given \( f(x) = x^2 + 1 \) on the interval \([-2, 2]\), check the hypotheses of Rolle’s Theorem and the Mean Value Theorem, and find a value \( c \) that makes the appropriate conclusion true.

Solution: As \( f'(x) = 2x \), \( f(x) \) is continuous on \([-2, 2]\) and differentiable on \((-2, 2)\). Note that \( f(-2) = 4 + 1 = f(2) \), and so the hypothesis of Rolle’s Theorem is satisfied.

To find \( c \) in \((-2, 2)\) such that \( f'(c) = 0 \), we solve the equation \( 2c = f'(c) = 0 \) to get \( c = 0 \).

Example 3 Show that the function \( f(x) = \sqrt{x - 1} \) satisfies the hypotheses of The Mean Value Theorem on \([2, 5]\), and find all numbers \( c \) in \((2, 5)\) that satisfy the conclusion of that theorem.

Solution: Since \( f(x) \) is a composition function of a power function \( (f(u) = \sqrt{u}) \) and a polynomial \( (u = x - 1) \), \( f(x) \) is continuous in its domain \([1, \infty)\), and differentiable in \((1, \infty)\); and in particular, \( f(x) \) is continuous on \([2, 5]\), and differentiable on \((2, 5)\). Thus \( f(x) \) satisfies the hypotheses of The Mean Value Theorem on \([2, 5]\).

Compute \( f(2) = 1 \) and \( f(5) = 2 \); and \( f'(x) = \frac{1}{2\sqrt{x-1}} \). As in this example, \( a = 2 \) and \( b = 5 \),

\[
\frac{1}{2\sqrt{x-1}} = f'(x) = \frac{f(b) - f(a)}{b - a} = \frac{2 - 1}{5 - 2} = \frac{1}{3}.
\]

Thus \( 2\sqrt{x-1} = 3 \), and so \( 4(x-1) = 9 \). It follows that \( x = \frac{13}{4} \), and so the only point \( c \) satisfying the conclusion of that theorem is \( c = \frac{13}{4} \).
Use Rolle’s Theorem to determine the number of solutions of an equation

**Facts:** Let \( f(x) \) be a function continuous on \([a, b]\) and differentiable in \((a, b)\).

1. If \( f(x) = 0 \) has two distinct solutions in \([a, b]\), then \( f'(x) = 0 \) has one solution in \((a, b)\).
2. More generally, for an integer \( n \geq 2 \), if \( f(x) = 0 \) has \( n \) distinct solutions in \([a, b]\), then \( f'(x) = 0 \) has \( n - 1 \) distinct solutions in \((a, b)\).

**Example 4** Show that \( x^3 + 4x - 3 = 0 \) has exactly one solution.

**Solution:** Let \( f(x) = x^3 + 4x - 3 \). As \( f(0) = -3 < 0 \) and \( f(1) = 2 > 0 \), and as \( f(x) \) is continuous, by the Intermediate Value Theorem for continuous functions, \( f(x) = 0 \) must have at least one solution.

As \( f'(x) = x^2 + 4 > 0 \), \( f(x) \) has at most one solution.

Determine if a function is increasing or decreasing

**Facts:** Let \( f(x) \) be a function on an interval \( I \).

1. If for any pair of points \( x_1, x_2 \) in \( I \) with \( x_1 < x_2 \) we always have \( f(x_1) > f(x_2) \) (respectively, \( f(x_1) < f(x_2) \)) then \( f(x) \) is decreasing (respectively, increasing) in the interval \( I \). If for any pair of points \( x_1, x_2 \) in \( I \) with \( x_1 < x_2 \) we always have \( f(x_1) \leq f(x_2) \) (respectively, \( f(x_1) \geq f(x_2) \)) then \( f(x) \) is non increasing (respectively, non decreasing) in the interval \( I \).
2. If \( f'(x) > 0 \) (respectively, \( f'(x) < 0 \)) for all \( x \) in \( I \), then \( f(x) \) is increasing (respectively, decreasing) in the interval \( I \).

**Example 5** Given \( f(x) = x^3 + 5x + 1 \), determine if \( f(x) \) is increasing, decreasing, or neither.

**Solution:** As \( f'(x) = 3x^2 + 5 > 0 \), \( f(x) \) is increasing.

**Example 6** Given \( f(x) = \ln x \), determine if \( f(x) \) is increasing, decreasing, or neither.

**Solution:** The domain of \( f(x) \) is \((0, \infty)\). As \( f'(x) = \frac{1}{x} > 0 \), \( f(x) \) is increasing in its domain.