Compute derivatives of implicit functions

**Facts:** Given an equation \( F(x, y) = 0 \) involving variables \( x \) and \( y \) which defines \( y \) as a function \( y = y(x) \), to compute \( y' = \frac{dy}{dx} \), we can apply the following procedure.

(Step 1) View \( y = y(x) \) and differentiate both sides of the equation \( F(x, y) = 0 \) with respect to \( x \) (often Chain Rule is needed here). This will yield a new equation involving \( x, y \) and \( y' \).

(Step 2) Solve the resulting equation from (Step 1) for \( y' \).

**Example 1** Given \( x^4 + x^2 y^2 + y^4 = 48 \), find \( \frac{dy}{dx} \).

**Solution:** View \( y = y(x) \) and differentiate both sides of the equation \( x^4 + x^2 y^2 + y^4 = 48 \) to get
\[
4x^3 + 2xyy' + 2x^2 yy' + 4y^3 y' = 0.
\]
To solve this new equation for \( y' \), we first combine those terms involving \( y' \),
\[
(2x^2 y + 4y^3)y' = -4x^3 - 2xy^2,
\]
and then solve for \( y' \):
\[
y' = \frac{-4x^3 - 2xy^2}{2x^2 y + 4y^3}.
\]

**Example 2** Find an equation of line tangent to the curve \( xy^2 + x^2 y = 2 \) at the point \((1, -2)\).

**Solution:** The slope \( m \) of this line, is \( \frac{dy}{dx} \) at \((1, -2)\), and so we need to find \( y' \) first. Apply implicit differentiation. We differentiate both sides of the equation \( xy^2 + x^2 y = 2 \) with respect to \( x \) (view \( y = y(x) \) in the process) to get
\[
y^2 + 2xyy' + 2xy + x^2 y' = 0.
\]
Then we solve for \( y' \). First we have \( (2xy + x^2)y' = -y^2 - 2xy \), and then
\[
y' = \frac{-y^2 - 2xy}{2xy + x^2}.
\]
At \((1, -2)\), we substitute \( x = 1 \) and \( y = -2 \) in \( y' \) to get the slope \( m = \frac{-(-2)^2 - 2(1)(-2)}{2(1)(-2) + 1^2} = 0 \), and so the tangent line is \( y = -2 \).

**Example 3** Find all the points on the graph of \( x^2 + y^2 = 4x + 4y \) at which the tangent line is horizontal.

**Solution:** First find \( y' \). We differentiate both sides of the equation \( x^2 + y^2 = 4x + 4y \) with respect to \( x \) (view \( y = y(x) \) in the process) to get
\[
2x + 2yy' = 4 + 4y'.
\]
Then we solve for \( y' \). First we have \( (2y - 4)y' = 4 - 2x \), and then
\[
y' = \frac{2 - x}{y - 2}.
\]
Note that when \( x = 2 \), the equation \( x^2 + y^2 = 4x + 4y \) becomes \( 4 + y^2 = 8 + 4y \), or \( y^2 - 4y = 4 \). Solve this equation we get \( y = 2 + \sqrt{8} \) and \( y = 2 - \sqrt{8} \). Therefore, at \((2, 2 - \sqrt{8})\) and \((2, 2 + \sqrt{8})\), the curve has horizontal tangent lines.
Compute related rates

The Problem: Given an equation \( F(x, y) = 0 \), where \( x = x(t) \) and \( y = y(t) \), and given values of \( x, y \) and one of \( x'(t) \) and \( y'(t) \) (say \( x'(t) \)), we want to find the missing rate value (in this case is \( y'(t) \)).

(Step 1) View \( x = x(t) \) and \( y = y(t) \), apply chain rule to differentiate both sides of the equation \( F(x, y) = 0 \) with respect to \( t \). This will yield a new equation involving \( x, y, x'(t) \) and \( y'(t) \).

(Step 2) Solve the resulting equation from (Step 1) for \( y'(t) \), assuming that the values of \( x, y \) and \( x'(t) \) are given.

Remark: Please distinguish this problem with the implicit differentiation problem.

Example 1  A circular oil slick of uniform thickness is caused by a spill of 1 m\(^3\) of oil. The thickness of the oil slick is decreasing at the rate of 0.1 cm/h. At what rate is the radius of the slick increasing when the radius is 8m?

Solution: Let \( r \) and \( h \) denote the radius and the thickness of the oil slick, respectively. Then both \( r = r(t) \) and \( h = h(t) \) are functions of the times \( t \). That the volume of the slick is 1m\(^3\) becomes

\[
\pi r^2 h = 1.
\]

View \( r = r(t) \) and \( h = h(t) \) and differentiating both sides of this equation with respect to \( t \), we get

\[
2\pi rr'h + \pi r^2 h' = 0.
\]

We shall use meter as the unit for length. Therefore, \( h'(t) = -0.001 \text{m/h} \). When \( r = 8 \), we have \( h = \frac{1}{8\pi} = \frac{1}{64\pi} \). Substitute all these in to equation involving the rates, we have

\[
2\pi(8)r'(t) \frac{1}{64\pi} + \pi64(-0.001) = 0, \text{ and so } r'(t) = \frac{4 \cdot 64\pi}{1000} = \frac{32\pi}{125} \text{m/h}.
\]

Thus when the radius is 8m, the radius of the slick increasing at the rate of \( \frac{32\pi}{125} \) m/h.

Example 2  The width of a rectangle is half its length. At what rate is its area increasing if its width is 10cm and is increasing at 0.5 cm/s\(^2\)?.

Solution: Let \( w \) and \( l \) denote the width and the length of the rectangle, respectively. Then both \( l = l(t) \) and \( w = w(t) \) are functions of the time \( t \). Moreover, \( 2w = l \). Thus the area \( A = lw = 2w^2 \).

View \( w = w(t) \) and differentiating \( A(t) \) with respect to \( t \), we get

\[
A'(t) = 4ww'(t).
\]

When \( w = 10 \text{cm} \) and \( w'(t) = 0.5 \text{ cm/s}^2 \), we have \( A'(t) = 4(10)(0.5) = 20 \text{ cm/s} \).