1. Given \( f(x) = \frac{4x^2 - x + 3}{\sqrt{x}} \), find \( f'(x) \) and \( f''(x) \).

2. Given \( f(x) = x^{10} + 4x^3 - 2x + 1 \), find \( f^{(5)}(x) \).

3. Find an equation of the line tangent to \( y = f(x) = x^3 + 2x - 10 \) at \( x = 2 \).

4. Given \( f(x) = (x^{10} + 4x^3 - 1)(\sqrt{x} + x^2) \), use product rule to compute \( f'(x) \).

5. Given \( f(x) = \frac{x^{10} + 4x^3 - 1}{\sqrt{x} + x^2} \), use quotient rule to compute \( f'(x) \).
6. Given \( f(x) = (\sin(x^3) + e^x)^{10} \), find \( f'(x) \).

5. Given \( f(x) = \sqrt[3]{\cos(x) + \ln(x^2)} \), find \( f'(x) \).

6. Given \( f(x) = \frac{1}{\sqrt[3]{\tan^2(x) + e^{x^2}}} \), find \( f'(x) \).

7. Given \( f(x) = \sin(\ln(\cot(x^3))) \), find \( f'(x) \).

8. Find an equation of the line tangent to \( f(x) = \sqrt{x^2 + 9} \) at \( x = 4 \).
Math 155 – Spring 2004 WORKSHEET 4, Solutions

1. Given \( f(x) = \frac{4x^2 - x + 3}{\sqrt{x}} \), find \( f'(x) \) and \( f''(x) \).

**Solutions:** Use Properties of Exponents and write

\[
f(x) = \frac{4x^2 - x + 3}{\sqrt{x}} = \frac{4x^2 - x + 3}{x^{1/2}} = 4x^{3/2} - x^{1/2} + 3x^{-1/2} = 4x^{3/2} - x^{1/2} + 3x^{-1/2}.
\]

Then compute the derivatives

\[
f'(x) = 4 \cdot \frac{5}{3} x^{\frac{3}{2}} - \frac{2}{3} x^{-\frac{1}{2}} + 3 \cdot \left(-\frac{1}{3}\right) x^{-\frac{3}{2}} = \frac{20}{3} x^{\frac{3}{2}} - \frac{2}{3} x^{-\frac{1}{2}} - x^{-\frac{3}{2}}
\]

\[
f''(x) = 20 \cdot \left(\frac{1}{3}\right) x^{\frac{1}{2}} - \frac{2}{3} \cdot \left(-\frac{4}{3}\right) x^{-\frac{5}{2}} - \frac{40}{9} x^{-\frac{7}{2}} + \frac{2}{9} x^{-\frac{9}{2}} + \frac{4}{3} x^{-\frac{11}{2}}
\]

\[
f'''(x) = \frac{40}{9} \cdot \left(-\frac{1}{3}\right) x^{-\frac{3}{2}} + \frac{2}{9} \cdot \left(-\frac{7}{3}\right) x^{-\frac{5}{2}} + \frac{4}{3} \cdot \left(-\frac{4}{3}\right) x^{-\frac{7}{2}} = -\frac{40}{27} x^{-\frac{3}{2}} - \frac{8}{27} x^{-\frac{5}{2}} - \frac{28}{9} x^{-\frac{9}{2}}
\]

2. Given \( f(x) = x^{10} + 4x^3 - 2x + 1 \), find \( f^{(5)}(x) \).

**Solutions:** Let \( g(x) = 4x^3 - 2x + 1 \). As the fifth derivative of any polynomial of degree at most 4 must be zero, \( g^{(5)}(x) = 0 \). Thus, it suffices to compute the 5th derivative of \( x^{10} \). Therefore,

\[
f'(x) = 10x^9 + g'(x), f''(x) = 90x^8 + g''(x), f'''(x) = 720x^7 + g'''(x), f^{(4)}(x) = 5040x^6 + g^{(4)}(x),
\]

\[
f^{(5)}(x) = 30240x^5.
\]

3. Find an equation of the line tangent to \( y = f(x) = x^3 + 2x - 10 \) at \( x = 2 \).

**Solutions:** When \( x = 2 \), \( y = f(2) = 2^3 - 2(2) - 10 = 6 \). As \( f'(x) = 3x^2 + 2 \), the slope of the line is \( f'(2) = 3 \cdot 2^2 + 2 = 14 \). An equation of this line is

\[y - 6 = 14(x - 2)\]

4. Given \( f(x) = (x^{10} + 4x^3 - 1)(\sqrt{x} + x^2) \), use product rule to compute \( f'(x) \).

**Solutions:** Write \( \sqrt{x} = x^{1/2} \) (so that we can use the power rule) and apply product rule to get

\[
f'(x) = \frac{10x^9 + 12x^2}{(\sqrt{x} + x^2)} (x^{10} + 4x^3 - 1)(\frac{1}{2x^{1/2}} + 2x).
\]

5. Given \( f(x) = \frac{x^{10} + 4x^3 - 1}{\sqrt{x} + x^2} \), use quotient rule to compute \( f'(x) \).

**Solutions:** Write \( \sqrt{x} = x^{1/2} \) (so that we can use the power rule) and apply quotient rule to get

\[
f'(x) = \frac{(10x^9 + 12x^2)\sqrt{x} + x^2 - (x^{10} + 4x^3 - 1)(\frac{1}{2\sqrt{x}} + 2x)}{(\sqrt{x} + x^2)^2}.
\]
6. Given \( f(x) = (\sin(x^3) + e^x)^{10} \), find \( f'(x) \).

Solutions: Let \( u(x) = \sin(x^3) + e^x \) and so \( f = u^{10} \). Apply the chain rule to compute

\[
\frac{du}{dx} = (\sin(x^3))' + (e^x)' = \cos(x^3) \cdot (3x^2) + e^x = 3x^2 \cos(x^3) + e^x.
\]

Apply the chain rule to get the answer:

\[
\frac{df}{dx} = 10u^9 \cdot \frac{du}{dx} = 10(\sin(x^3) + e^x)^9(3x^2 \cos(x^3) + e^x).
\]

7. Given \( f(x) = \sqrt[3]{\cos(x) + \ln(x^2)} \), find \( f'(x) \).

Solutions: Write the function in exponential form (and we may use properties of logarithm to write \( \ln(x^2) = 2\ln(x) \)) to get \( f(x) = (\cos(x) + 2\ln(x))^{\frac{1}{3}} \). Then apply chain rule (view \( u = \cos(x) + 2\ln(x) \) and \( f = u^{\frac{1}{3}} \))

\[
f'(x) = \frac{1}{3}(\cos(x) + 2\ln(x))^{-\frac{2}{3}} \left( -\sin(x) + \frac{2}{x} \right).
\]

8. Given \( f(x) = \frac{1}{\sqrt[3]{\tan^2(x) + e^{x^2}}} \), find \( f'(x) \).

Solutions: Write the function in exponential form \( f(x) = (\tan^2(x) + e^{x^2})^{-\frac{1}{3}} \). Then apply chain rule (view \( u = \tan^2(x) + e^{x^2} \) and \( f = u^{-\frac{1}{3}} \))

\[
f'(x) = -\frac{1}{3}(\tan^2(x) + e^{x^2})^{-\frac{4}{3}} \left( 2\tan(x) \sec^2(x) + 2xe^{x^2} \right).
\]

9. Given \( f(x) = \sin(\ln(\cot(x^3))) \), find \( f'(x) \).

Solutions: This function is a composition of several functions. Apply the chain rule one step after another.

\[
f'(x) = \cos(\ln(\cot(x^3))) \cdot \frac{1}{\cot(x^3)} \cdot (\sin(x^3)) \cdot (3x^2).
\]

10. Find an equation of the line tangent to \( f(x) = \sqrt{x^2 + 9} \) at \( x = 4 \).

Solutions: When \( x = 4 \), \( y = f(4) = \sqrt{16 + 9} = 5 \). Compute \( f'(x) = \frac{1}{2\sqrt{x^2 + 9}} \cdot (2x) = \frac{x}{\sqrt{x^2 + 9}} \). The slope of the line is \( f'(4) = \frac{4}{5} \). Therefore, the line has equation

\[
y - 5 = \frac{4}{5}(x - 4).
\]