1. Find the value of the constant $c$ such that the resulting function is continuous if

$$f(x) = \begin{cases} 
5x - c, & \text{if } x \leq 1 \\
3cx^2 + 1, & \text{if } x > 1
\end{cases}$$

2. Find all the discontinuities of the function

$$f(x) = \begin{cases} 
4x + 1 & \text{if } x < -1 \\
x^2 + 1 & \text{if } -1 \leq x < 0 \\
\tan \frac{x}{x} & \text{if } x > 0
\end{cases}$$

determine which discontinuity is removable and indicate how to remove it if there is one.

3. Compute $f'(a)$, where $f(x) = \frac{1}{x+2}$, at $a = 1$ by applying the definition of a derivative.
4. Find an equation of the line tangent to the curve \( y = 4 - 7x + x^2 \) at \( x = 2 \).

5. Compute the derivative of \( f(x) = \frac{x^5 + x^3 - \sqrt{x}}{x^2} \).

6. Compute the derivative of \( f(x) = (x^3 + x)(2x + 1) \).

7. Compute the derivative of \( f(x) = \frac{1}{\sqrt{x + 2}} \) by applying the definition of a derivative.
1. Find the value of the constant $c$ such that the resulting function is continuous if

$$f(x) = \begin{cases} 
5x - c, & \text{if } x \leq 1 \\
3x^2 + 1, & \text{if } x > 1 
\end{cases}$$

**Solution:** Since $f(x)$ is a polynomial $5x - c$ in $(-\infty, 1)$ and a polynomial $3x^2 + 1$ in $(1, \infty)$, $f(x)$ is continuous on both open intervals. The only point which we are not sure is at $x = 1$.

Compute the side limits

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 5x - c = 5 - c,$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 3x^2 + 1 = 3c + 1.$$ 

In order for $f(x)$ to be continuous at $x = 1$, both side limits must equal, and so we must have $5 - c = 3c + 1$. It follows that $c = 1$.

Answer: we need to have $c = 1$ so that the function will be continuous.

2. Find all the discontinuities of the function

$$f(x) = \begin{cases} 
4x + 1 & \text{if } x < -1 \\
x^2 + 1 & \text{if } -1 \leq x < 0 \\
\tan x & \text{if } x > 0 
\end{cases}$$

determine which discontinuity is removable and indicate how to remove it if there is one.

**Solution:** As $4x + 1, x^2 + 1$ and $\tan x$ are continuous functions in their domains, the only candidate of discontinuities are $x = -1$ and $x = 0$.

At $x = -1$,

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} 4(-1) + 1 = -3,$$
$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (-1)^2 + 1 = 2.$$

As the two side limits do not agree, the limit $\lim_{x \to -1} f(x)$ does not exist. Therefore, $x = -1$ is a non removable discontinuity.

At $x = 0$, as 0 is not in the domain, $x = 0$ is a discontinuity of $f(x)$. Note that

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0^2 + 1 = 1,$$
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\tan x}{x} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\sin x}{x \cos x} = 1.$$ 

Therefore, the limit $\lim_{x \to 0} f(x) = 1$ exists. Therefore, $x = 0$ is a removable discontinuity, and it can be removed by redefining $f(0) = 1$.

3. Compute $f'(a)$, where $f(x) = \frac{1}{x + 2}$, at $a = 1$ by applying the definition of a derivative.

**Solution:** Compute $f(1 + h) = \frac{1}{(1 + h) + 2} = \frac{1}{3 + h}$ and $f(1) = \frac{1}{1 + 2} = \frac{1}{3}$.

Compute (and cancel the zero factor $h$)

$$\frac{f(1 + h) - f(1)}{h} = \frac{1}{h} \left( \frac{1}{3 + h} - \frac{1}{3} \right) = \frac{1}{h} \left( \frac{3 - (3 + h)}{3(3 + h)} \right) = \frac{-h}{3(3 + h)} = \frac{-1}{3(3 + h)}.$$ 

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Apply the definition to get
\[ f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{-1}{3(3 + h)} = \frac{-1}{9}. \]

4. Find an equation of the line tangent to the curve \( y = 4 - 7x + x^2 \) at \( x = 2 \).

**Solution:** First we compute the derivative of \( y = 4 - 7x + x^2 \) at \( x = 2 \), which will be the slope \( m \) of the tangent line. Let \( f(x) = 4 - 7x + x^2 \). Then \( f(2 + h) = 4 - 7(2 + h) + (2 + h)^2 \), \( f(2) = 4 - 7(2) + 2^2 \), and
\[
\frac{f(2 + h) - f(2)}{h} = \frac{-7h + 4h + h^2}{h} = -7 + 4 + h = -3 + h.
\]
Thus
\[
m = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} -3 + h = -3.
\]
It follows an equation of the line is
\[ y - (-6) = (-3)(x - 2). \]

5. Compute the derivative of \( f(x) = \frac{x^5 + x^3 - \sqrt{x}}{x^2} \).

**Solution:** Write \( f(x) = \frac{x^5 + x^3 - \sqrt{x}}{x^2} = x^3 + x - x^{-\frac{3}{2}} \). Thus
\[ f'(x) = 3x^2 + 1 - \frac{-3}{2}x^{-\frac{5}{2}}. \]

6. Compute the derivative of \( f(x) = (x^3 + x)(2x + 1) \).

**Solution:** Write \( f(x) = (x^3 + x)(2x + 1) = 2x^4 + x^3 + 2x^2 + x \). Thus
\[ f'(x) = 8x^3 + 3x^2 + 4x + 1. \]

7. Compute the derivative of \( f(x) = \frac{1}{\sqrt{x + 2}} \) by applying the definition of a derivative.

**Solution:** Compute \( f(x + h) = \frac{1}{\sqrt{(x+h)+2}} \).
Compute (and cancel the zero factor $h$)

\[
\frac{f(x + h) - f(x)}{h} = \frac{1}{h} \left( \frac{1}{\sqrt{x + h + 2}} - \frac{1}{\sqrt{x + 2}} \right)
\]

\[
= \frac{1}{h} \left( \frac{\sqrt{x + 2} - \sqrt{x + h + 2}}{\sqrt{x + h + 2} \cdot \sqrt{x + 2}} \right)
\]

\[
= \frac{1}{h} \left( \frac{\sqrt{x + 2} - \sqrt{x + h + 2}}{\sqrt{x + h + 2} \cdot \sqrt{x + 2}} \right) \left( \frac{\sqrt{x + 2} + \sqrt{x + h + 2}}{\sqrt{x + h + 2} + \sqrt{x + 2}} \right)
\]

\[
= \frac{1}{h} \left( \frac{(x + 2) - (x + h + 2)}{\sqrt{x + h + 2} \cdot \sqrt{x + 2} \cdot (\sqrt{x + h + 2} + \sqrt{x + 2})} \right)
\]

\[
= \frac{1}{h} \left( \frac{-h}{\sqrt{x + h + 2} \cdot \sqrt{x + 2} \cdot (\sqrt{x + h + 2} + \sqrt{x + 2})} \right)
\]

\[
= \frac{-1}{\sqrt{x + h + 2} \cdot \sqrt{x + 2} \cdot (\sqrt{x + h + 2} + \sqrt{x + 2})}
\]

Apply the definition to get

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{-1}{\sqrt{x + h + 2} \cdot \sqrt{x + 2} \cdot (\sqrt{x + h + 2} + \sqrt{x + 2})}
\]

\[
= \frac{-1}{\sqrt{x + 0 + 2} \cdot \sqrt{x + 2} \cdot (\sqrt{x + 0 + 2} + \sqrt{x + 2})}
\]

\[
= \frac{-1}{2(x + 2)\sqrt{x + 2}}
\]