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VECTOR FLOWS AND INTEGER FLOWS*

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Abstract. A vector S^d-flow is a flow whose flow values are vectors in S^d, where S^d is the set of all unit vectors in $\mathbb{R}^{d+1}$. Jain [*Open Problem Garden*, http://www.openproblemgarden.org/op/unit_vector_flows (2007)] and Thomassen [*J. Combin. Theory Ser. B.*, 108 (2014), pp. 81–91] proved that a graph has a vector S^1-flow if it has a nowhere-zero integer 3-flow. Thomassen [*J. Combin. Theory Ser. B.*, 108 (2014), pp. 81–91] pointed out that a graph admitting a vector S^1-flow may not necessarily admit a nowhere-zero integer 3-flow and presented a family of examples showing that the converse is not true. The rank of a vector S^1-flow $(D, \boldsymbol{f})$ is defined as the rank of linear space generated by all balanced vectors $\boldsymbol{\epsilon}(v) = (\epsilon_1(v), \epsilon_2(v), \ldots, \epsilon_b(v))$ for all $v \in V(G)$, where $\epsilon_i(v)$ is the difference between the number of outgoing edges with flow value $\boldsymbol{\alpha}_i$ from v and the number of ingoing edges with the same flow value to v. In this paper, we prove that G admits a nowhere-zero integer 3-flow if G admits a vector S^1-flow with rank at most two. This result is sharp since there are examples that admit vector S^1-flows with rank at least 3, but no nowhere-zero integer 3-flows.

Key words. integer flow, vector flow, vector S^1-flow, modulo k-orientation

AMS subject classifications. 05C10, 05C15

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1. Introduction.

Graphs in this paper may have multiple edges and loops. We follow the notation and terminology of [1] except otherwise stated.

1.1. Integer flows.

DEFINITION 1.1. *Let G be a graph with an orientation D and k be a natural number. The ordered pair (D, f) is a nowhere-zero integer k-flow if $f : E(G) \mapsto \{\pm 1, \ldots, \pm(k-1)\}$ such that, for every vertex $v \in V(G)$,*

$$\sum_{e \in E^+(v)} f(e) \;-\; \sum_{e \in E^-(v)} f(e) = 0.$$

The concept of integer flows was introduced by Tutte as a dual of the vertex coloring problem. The following is one of the major open problems in graph theory.

CONJECTURE 1.2. (Tutte [7], 3-flow conjecture) *Every 4-edge-connected graph admits a nowhere-zero integer 3-flow.*

Extended from a recent breakthrough in [5], the 3-flow conjecture (Conjecture 1.2) is verified for all odd-7-edge-connected graphs [4].

THEOREM 1.3. *Every odd-7-edge-connected graph admits a nowhere-zero integer 3-flow.*

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1.2. Vector flows.

DEFINITION 1.4. *Let $\mathbb{U}$ be a subset of vectors in the Euclidian space $\mathbb{R}^n$, and G be a graph with an orientation D. An ordered pair $(D, \boldsymbol{f})$ is called a vector $\mathbb{U}$-flow of G if $\boldsymbol{f} : E(G) \mapsto \mathbb{U}$ such that, for every vertex $v \in V(G)$,*

$$\sum_{e \in E^+(v)} \boldsymbol{f}(e) \; - \sum_{e \in E^-(v)} \boldsymbol{f}(e) = \mathbf{0}. \tag{1}$$

CONJECTURE 1.5 (Jain [3]). *Every 4-edge-connected graph admits a vector S^1-flow where S^1 is the set of all vectors $\boldsymbol{\alpha}$ in $\mathbb{R}^2$ with $\|\boldsymbol{\alpha}\| = 1$.*

Following [6], the goal of this paper is to study the relation between the two graphic properties: integer 3-flows and vector S^1-flows.

Statement I. A graph G admits a nowhere-zero integer 3-flow.

Statement II. A graph G admits a vector S^1-flow.

Statement III. A graph G admits a vector C_3-flow, where C_3 is the set of three complex roots of the unit.

It was asked by DeVos [3] whether Statements I and II are equivalent. The following theorem responds to this question and unveils some relations between these two graphic properties.

THEOREM 1.6 (Thomassen [6]).

(1) *Statements* I *and* III *are equivalent.*

(2) *Statement* I *implies Statement* II, *but not vise versa.*

(3) *Statements* I *and* II *are equivalent for cubic graphs.*

Thomassen [6] discovered a family of graphs that admit vector S^1-flows (Statement II) but no nowhere-zero integer 3-flows (Statement I). See section 2 for the detailed discussion of this family of counterexamples.

Due to the existence of such counterexamples, it is natural to consider the following problem.

PROBLEM 1.7. *Characterize vector S^1-flows for which Statement* II *implies Statement* I. *That is, if G admits a vector S^1-flow (not necessary a vector C_3-flow) with certain properties, then G also admits a nowhere-zero integer* 3*-flow.*

The vector C_3-flow (described in Statement III) is one of such examples for which Statement II implies Statement I.

PROBLEM 1.8. *Characterize graphs for which Statement* II *implies Statement* I. *That is, if G is a graph with certain properties and admits a vector S^1-flow, then G also admits a nowhere-zero integer* 3*-flow.*

Let $(D, \boldsymbol{f})$ be a vector S^1-flow of a graph G with flow values $\{\boldsymbol{f}(e) : e \in E(G)\} = \{\pm\boldsymbol{\alpha}_1, \pm\boldsymbol{\alpha}_2, \dots, \pm\boldsymbol{\alpha}_b\}$, where the set $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_b\}$ consists of b pairwise linearly independent vectors of S^1. We may further assume $\{\boldsymbol{f}(e) : e \in E(G)\} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_k\}$, otherwise we reverse the orientations of all edges with flow values $-\boldsymbol{\alpha}_i$ and negate their flow values. We call an edge an $\boldsymbol{\alpha}_i$-edge of G if this edge is assigned vector flow $\boldsymbol{\alpha}_i$, and define the family of all $\boldsymbol{\alpha}_i$-edges by $E_i(G) = \{e \in E(G) : \boldsymbol{f}(e) = \boldsymbol{\alpha}_i\}$ for each $i = 1, 2, \dots, b$.

DEFINITION 1.9. *Let $(D, \boldsymbol{f})$ be a vector S^1-flow of a graph G and $\{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_b\}$ be defined as above. For each vertex $v \in V(G)$, denote*

$$\boldsymbol{\epsilon}(v) = \langle \epsilon_1(v), \dots, \epsilon_b(v) \rangle,$$

where $\epsilon_i(v) = \epsilon_i^+(v) - \epsilon_i^-(v)$, $\epsilon_i^+(v) = |E^+(v) \cap E_i(G)|$, and $\epsilon_i^-(v) = |E^-(v) \cap E_i(G)|$ for each $i = 1, 2, \dots, b$. The vector $\boldsymbol{\epsilon}(v)$ is called the balanced vector at vertex v. A vertex v is called trivial if $\boldsymbol{\epsilon}(v) = \mathbf{0}$, and is called unsplittable if, for each i, either

$\epsilon_i^+(v)$ or $\epsilon_i^-(v)$ is zero. The balanced equation of the vertex v is defined as

$$\epsilon_1(v)\boldsymbol{\alpha}_1 + \epsilon_2(v)\boldsymbol{\alpha}_2 + \cdots + \epsilon_b(v)\boldsymbol{\alpha}_b = \mathbf{0}. \tag{2}$$

Let $S(\boldsymbol{f})$ denote the linear subspace of $\mathbb{R}^b$ generated by all balanced vectors of $(D, \boldsymbol{f})$ of G. And the rank of the subspace $S(\boldsymbol{f})$, denoted by rank$(\boldsymbol{f})$, is also called the rank of the S^1-flow $\boldsymbol{f}$.

The following is one of our main theorems, which is a partial solution to Problem 1.7.

THEOREM 1.10. *If a graph G admits a vector S^1-flow with rank at most two, then G admits a nowhere-zero integer 3-flow (that is, Statement* II *implies Statement* I *if the rank of vector S^1-flow is at most two).*

Note that all the vector C_3-flows (in Statement III) are vector S^1-flows with rank one. Theorem 1.10, in some sense, is an extension of Theorem 1.6(1) and is the study of the converse of Theorem 1.6(2).

Theorem 1.10 is sharp in the sense that there are graphs that admit vector S^1-flows $(D, \boldsymbol{f})$ with rank three, but no nowhere-zero integer 3-flows (discovered by Thomassen [6]; see section 2).

In fact, we prove a stronger result as follows.

THEOREM 1.11. *If a graph G admits a vector S^1-flow with rank at most two, then either G is Eulerian and thus admits a nowhere-zero integer 2-flow, or G admits a circular $(2+\frac{1}{p})$-flow for some integer $p \geq 1$.*

The following result is motivated by Problem 1.8 and extends Theorem 1.6(3).

THEOREM 1.12. *Let G be a graph and V_3 be the set of vertices of degree 3 in G. If $G[V_3]$ is connected and $G - V_3$ is acyclic, then Statements* I *and* II *are equivalent (that is, G admits a nowhere-zero integer 3-flow if and only if G admits a vector S^1-flow).*

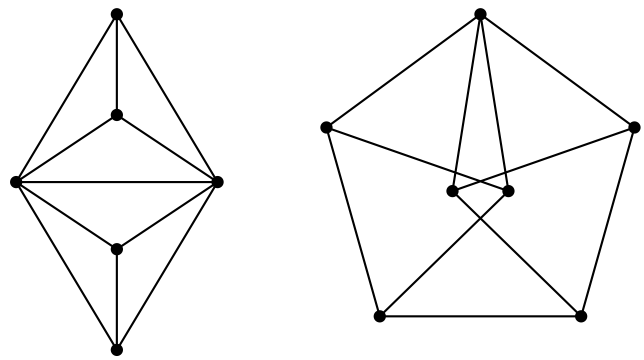
Remark. Thomassen [6] found a family of graphs that admit vector S^1-flows but no nowhere-zero integer 3-flows. See Figure 1(a) and note that $G[V_3]$ has two components. Thus the condition that $G[V_3]$ is connected in Theorem 1.12 may not be dropped.

2. Vector S^1-flows of the duals of unit distance graphs. In [6], Thomassen gave a characterization of planar graphs admitting vector S^1-flows as follows.

LEMMA 2.1 (Thomassen [6]). *Let G be a planar graph. Then G admits a vector S^1-flow if and only if the dual graph G^* is homomorphic to a subgraph of a unit distance graph, where a unit distance graph is a graph whose vertices are points in Euclidian plane $\mathbb{R}^2$ such that two vertices are adjacent if and only if they have distance 1.*

Based on Lemma 2.1, Thomassen [6] constructed a family of graphs that admit vector S^1-flows but no nowhere-zero integer 3-flows. One of them, denoted by G, is illustrated in Figure 1(a). The graph G admits a vector S^1-flow since the dual of G is homomorphic to a unit distance graph G^* (see Figure 1(b)). With no confusion and slight abuse of notation, let G^* be the dual of G. Note that G^* is the Hajós graph [1] with chromatic number 4. By Tutte's theorem [7], G admits a nowhere-zero integer 4-flow, but no nowhere-zero integer 3-flows. In fact, G^* can be obtained from two K_4 by Hajós construction. Repeatedly adding K_4 to G^* by Hajós construction, we can construct an infinite family of counterexamples.

Since G^* is a unit distance graph, each edge of G^* can be considered as a unit vector under some orientation D^*. It is easy to see that there are 7 pairwise linearly independent unit vectors among 11 edges of G^*. If we associate the vectors and orientations of edges in G^* with their dual edges in G, respectively, then we obtain



(a) G (b) G^*

FIG. 1. *Thomassen's counterexample G.*

a vector S^1-flow of G with orientation D. From Figure 1(b), one can easily see that the following is the set of all balanced vectors of the vector S^1-flow,

$$\{\pm\langle 1,1,1,0,0,0,0\rangle,\ \pm\langle 0,0,0,0,1,1,1\rangle,\ \pm\langle 1,1,0,1,0,1,1\rangle\}.$$

Thus, the rank of the vector flow is three since they are linearly independent.

Inspired by Theorem 1.10 and the above discussion related to the counterexample, we propose the following problem.

PROBLEM 2.2. *Find an integer r such that if a graph G admits a vector S^1-flow $(D, \boldsymbol{f})$ with rank at most rank r, then G admits a nowhere-zero integer 4-flow.*

3. Proof of Theorem 1.10.

3.1. Vector S^1-flows with rank one.

DEFINITION 3.1. *Let G be a graph and k be an odd integer $(k \geq 3)$. An orientation D of G is called a modulo k-orientation if, for each vertex $v \in V(G)$,*

$$d_D^+(v) \equiv d_D^-(v) \pmod{k}.$$

DEFINITION 3.2. *Let k and d be two integers with $0 < d \leq \frac{k}{2}$. A circular $\frac{k}{d}$-flow of a graph G is an integer flow (D, f) such that*

$$f : E(G) \longrightarrow \{\pm d, \pm(d+1), \ldots, \pm(k-d)\}.$$

PROPOSITION 3.3. (Jaeger [2]) *For any natural number p, a graph G admits a circular $(2+\frac{1}{p})$-flow if and only if G has a modulo $(2p+1)$-orientation.*

DEFINITION 3.4 (vertex splitting). *Let v be a vertex of G and $A = \{e_1, e_2, \ldots, e_t\} \subseteq E(v)$. The graph $G_{[v;A]} = G_{[v;e_1,e_2,\ldots,e_t]}$ is obtained from G by splitting v into two vertices v and w, where w is an end vertex of all the edges in A and v is an end vertex of the remaining edges in $E(v) \setminus A$. If $|A| = 2$ or $|E(v) \setminus A| = 2$, we further suppress the new degree 2 vertex w or v.*

The following lemma is a direct consequence of Definition 3.4.

LEMMA 3.5. *Suppose that G admits a vector S^1-flow $(D, \boldsymbol{f})$. Let $v \in V(G)$ and $A \subseteq E(v)$ such that*

$$\sum_{e \in A \cap E_D^+(v)} \boldsymbol{f}(e) = \sum_{e \in A \cap E_D^-(v)} \boldsymbol{f}(e).$$

Then

(1) *$G_{[v;A]}$ admits a vector S^1-flow $(D_A, \boldsymbol{f}_A)$ and $S(\boldsymbol{f}_A) = S(\boldsymbol{f})$;*

(2) *if $G_{[v;A]}$ has a modulo k-orientation, then G has a modulo k-orientation.*

DEFINITION 3.6 (elementary balanced vector). *Let $(D, \boldsymbol{f})$ be a vector S^1-flow of a graph G, where $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_b$ are the flow values in S^1 and rank $(\boldsymbol{f}) \leq 1$. Let $\boldsymbol{\epsilon} = \langle \epsilon_1, \ldots, \epsilon_b \rangle$ be defined as follows: either $\boldsymbol{\epsilon} = \mathbf{0}$ when $S(\boldsymbol{f}) = \{\mathbf{0}\}$ or $\boldsymbol{\epsilon}$ is a nonzero vector in $S(\boldsymbol{f})$ such that*

(1) *all coordinates of $\boldsymbol{\epsilon}$ are integers;*

(2) *subject to (1), $\|\boldsymbol{\epsilon}\| = \sum_{i=1}^{b} |\epsilon_i|$ is as small as possible.*

The vector $\boldsymbol{\epsilon}$ is called the elementary balanced vector of $(D, \boldsymbol{f})$. Clearly, $\boldsymbol{\epsilon}$ is unique up to scalar multiplication of -1. And its length is defined by

$$k^* = \|\boldsymbol{\epsilon}\| = \sum_{i=1}^{b} |\epsilon_i| \, .$$

In the following sections, for simplicity of the context, we denote $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_b\}$ the set of flow values of a vector S^1-flow of a graph G. If $(D, \boldsymbol{f})$ is a vector flow with rank one, we denote $\boldsymbol{\epsilon} = \langle \epsilon_1, \epsilon_2, \ldots, \epsilon_b \rangle$ the elementary balanced vector and denote k^* the length of $\boldsymbol{\epsilon}$.

LEMMA 3.7. *Suppose that G admits a vector S^1-flow $(D, \boldsymbol{f})$ with rank at most one. If G has at least one nontrivial vertex, then*

(1) *$k^* \geq 3$;*

(2) *for each vertex v, either v is unsplittable and $d(v)$ is a multiple of k^*, or v is splittable. Furthermore, if v is unsplittable, then v can be split into $\frac{d(v)}{k^*}$ vertices such that the balanced vector of each new vertex is either $\boldsymbol{\epsilon}$ or $-\boldsymbol{\epsilon}$.*

Proof.

(1) Since G has at least one nontrivial vertex, we have $\boldsymbol{\epsilon} \neq \mathbf{0}$. Note that $k^* \neq 1, 2$ by (2) and Definition 3.6. Thus $k^* \geq 3$.

(2) It suffices to prove that each unsplittable vertex v has the balanced vector $\boldsymbol{\epsilon}(v) = s\boldsymbol{\epsilon}$ for some integer s. Since $(D, \boldsymbol{f})$ has rank one, for each $v \in V(G)$, there exists some $s \in \mathbb{Q}$ such that $\boldsymbol{\epsilon}(v) = s\boldsymbol{\epsilon}$. Suppose to the contrary that $s = \frac{p}{q}$, where p, q are integers, $|q| > 1$, and $\gcd(p, q) = 1$. Then there exist two integers a, b such that $p = aq + b$ and $0 < |b| < |q|$. Thus $\frac{b}{q}\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(v) - a\boldsymbol{\epsilon}$ is an integer-valued vector but has smaller length than $\boldsymbol{\epsilon}$, which contradicts the definition of $\boldsymbol{\epsilon}$. ∎

Let G be a graph admitting a vector S^1-flow $(D, \boldsymbol{f})$ with rank one. If G contains some splittable nontrivial vertex v (that is, there exists at least one pair of edges incident with v with the same flow value but opposite orientations), then one can repeatedly vertex-split all possible pairs of edges at vertex v and other splittable vertices until there are no more splittable vertices. Let G' be the resulting graph. By Lemma 3.7(2), for each $v \in V(G')$, one can split v into $\frac{d_{G'}(v)}{k^*}$ vertices such that each new vertex has degree k^* and has the balanced vector either $\boldsymbol{\epsilon}$ or $-\boldsymbol{\epsilon}$. Denote the resulting graph by H.

DEFINITION 3.8. *Let G be a graph admitting a vector S^1-flow $(D, \boldsymbol{f})$ with rank one. If G contains at least one nontrivial vertex, then we call the graph H (constructed as above) the core of G.*

The following is a straightforward observation of Lemma 3.5 and Definition 3.8.

PROPOSITION 3.9. *Suppose that G admits a vector S^1-flow $(D, \boldsymbol{f})$ with rank one and that H is the core of G. Then*

(1) *H is a k^*-regular bipartite graph with vertex bipartition (A, B), where*

$$A = \{v \in V(H) : \boldsymbol{\epsilon}(v) = \boldsymbol{\epsilon}\},\ B = \{v \in V(H) : \boldsymbol{\epsilon}(v) = -\boldsymbol{\epsilon}\};$$

(2) *H admits a vector S^1-flow, say $(D_h, \boldsymbol{f}_h)$, and $S(\boldsymbol{f_h}) = S(\boldsymbol{f})$;*

(3) *H has a modulo k^*-orientation and therefore G has a modulo k^*-orientation;*

(4) *for each $i = 1, 2, \ldots, b$, $E_i(H)$ can be decomposed into $|\epsilon_i|$ perfect matchings of H, denoted by $M_1, M_2, \ldots, M_{|\epsilon_i|}$.*

Remark. Directing all the edges of H from A to B, the resulting orientation is a modulo k^*-orientation of H. Meanwhile, $E_i(H)$ has a perfect matching decomposition since the subgraph induced by $E_i(G)$ is an $|\epsilon_i|$-regular bipartite spanning subgraph of H.

THEOREM 3.10. *If a graph G admits a vector S^1-flow with rank at most one, then either G is Eulerian and thus admits a nowhere-zero integer 2-flow, or G admits a circular $(2 + \frac{1}{p})$-flow for some integer p with $k^* = 2p + 1$ and $p \geq 1$.*

Proof. Let $(D, \boldsymbol{f})$ be a vector S^1-flow with rank at most one. If $k^* = 0$, then v is trivial and $d_G(v)$ is even for each $v \in V(G)$ implying that G is Eulerian.

Now we assume $k^* > 0$. Then G contains at least one nontrivial vertex and thus G admits a modulo k^*-orientation by Proposition 3.9(3). If k^* is even, then G is Eulerian and admits a nowhere-zero integer 2-flow. Otherwise, by Proposition 3.3, G admits a circular $(2 + \frac{1}{p})$-flow with $k^* = 2p + 1$. ☐

3.2. Some useful lemmas. In this subsection, we present some lemmas which will be applied in the proof of the case of vector S^1-flows with rank two.

LEMMA 3.11. *Let $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_b\}$ be a set of b pairwise linearly independent vectors in S^1. Then the following statements hold.*

(1) *If $\sum_{i=1}^{b} \epsilon_i \boldsymbol{\alpha}_i = \mathbf{0}$ for some nonzero integers $\epsilon_1, \ldots, \epsilon_b$, then*

  (a) *for each j with $1 \leq j \leq b$,*

$$|\epsilon_j| < \sum_{i \neq j} |\epsilon_i|.$$

  (b) $\sum_{i=1}^{b} |\epsilon_i| \neq 1, 2, 4.$

(2) *If $\boldsymbol{\alpha}_1 + t\boldsymbol{\alpha}_2 + t\boldsymbol{\alpha}_3 = \mathbf{0}$ for some integer t, then for any nonzero integers $\epsilon_2, \epsilon_3, \epsilon_4$ with $|\epsilon_4| = 1$, we have*

$$\epsilon_2 \boldsymbol{\alpha}_2 + \epsilon_3 \boldsymbol{\alpha}_3 + \epsilon_4 \boldsymbol{\alpha}_4 \neq \mathbf{0}.$$

Proof. (1a) Note that $b \geq 3$ since each ϵ_i is nonzero. For each j with $1 \leq j \leq b$, we have $\epsilon_j \boldsymbol{\alpha}_j = -\sum_{i \neq j} \epsilon_i \boldsymbol{\alpha}_i$. So

$$|\epsilon_j| = |\epsilon_j \boldsymbol{\alpha}_j| = \left| \sum_{i \neq j} \epsilon_i \boldsymbol{\alpha}_i \right| \leq \left| \sum_{i \neq j} \right| \epsilon_i|,$$

with equality holding only if all $\boldsymbol{\alpha}_i$s are pairwise linearly dependent. Thus $|\epsilon_j| < \sum_{i \neq j} |\epsilon_i|$.

(1b) Obviously, $\sum_{i=1}^{b} |\epsilon_i| \neq 1, 2$. Suppose to the contrary that $\sum_{i=1}^{b} |\epsilon_i| = 4$. By (1a), $2|\epsilon_j| < \sum_{i=1}^{b} |\epsilon_i| = 4$ and therefore $|\epsilon_j| \leq 1$ for each $j = 1, \ldots, b$. Without loss of generality, we assume $|\epsilon_i| = 1$ for each $i = 1, 2, 3, 4$ and $\epsilon_j = 0$ for each $j \geq 5$. Then

$$\epsilon_1 \boldsymbol{\alpha}_1 + \epsilon_2 \boldsymbol{\alpha}_2 + \epsilon_3 \boldsymbol{\alpha}_3 + \epsilon_4 \boldsymbol{\alpha}_4 = \mathbf{0}.$$

This implies that the vectors $\epsilon_1 \boldsymbol{\alpha}_1, \epsilon_2 \boldsymbol{\alpha}_2, \epsilon_3 \boldsymbol{\alpha}_3$ and $\epsilon_4 \boldsymbol{\alpha}_4$ form a parallelogram in $\mathbb{R}^2$, which contradicts the assumption that they are pairwise linearly independent vectors.

(2) Suppose to the contrary that $\epsilon_2\boldsymbol{\alpha}_2+\epsilon_3\boldsymbol{\alpha}_3+\epsilon_4\boldsymbol{\alpha}_4=\mathbf{0}$. Since $\epsilon_4=\pm 1$, we have $|\epsilon_2|=|\epsilon_3|$ by (a). If $\epsilon_2=-\epsilon_3$, then by equations $\boldsymbol{\alpha}_1+t\boldsymbol{\alpha}_2+t\boldsymbol{\alpha}_3=\mathbf{0}$ and $\epsilon_2\boldsymbol{\alpha}_2-\epsilon_2\boldsymbol{\alpha}_3+\epsilon_4\boldsymbol{\alpha}_4=\mathbf{0}$, we have

$$\epsilon_2\boldsymbol{\alpha}_1+2t\epsilon_2\boldsymbol{\alpha}_2+t\epsilon_4\boldsymbol{\alpha}_4=\mathbf{0}.$$

Since $|\epsilon_2|\geq 1, |t|\geq 1$, and $|\epsilon_4|=1$, we have

$$|2t\epsilon_2|=|t\epsilon_2|+|t\epsilon_2|\geq|\epsilon_2|+|t\epsilon_4|,$$

which contradicts (a).

If $\epsilon_2=\epsilon_3$, then by equations $\boldsymbol{\alpha}_1+t\boldsymbol{\alpha}_2+t\boldsymbol{\alpha}_3=\mathbf{0}$ and $\epsilon_2\boldsymbol{\alpha}_2+\epsilon_2\boldsymbol{\alpha}_3+\epsilon_4\boldsymbol{\alpha}_4=\mathbf{0}$, we have

$$\epsilon_2\boldsymbol{\alpha}_1+t\epsilon_4\boldsymbol{\alpha}_4=\mathbf{0},$$

implying that $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_4$ are linearly dependent, a contradiction. ∎

LEMMA 3.12. *Suppose that a graph G admits a vector S^1-flow $(D,\boldsymbol{f})$ with rank at most one. If $k^*\leq 5$, then G has a modulo 3-orientation D' such that either $D'=D$ or D' can be obtained from D by reversing the orientations of all edges of some $E_i(G)$'s.*

Proof. If $k^*=0$, let $D'=D$ and then D itself is a modulo 3-orientation of G. Otherwise, G contains at least one nontrivial vertex and $k^*=3$ or 5 by Lemma 3.11(b). Let H be the core of G (see Definition 3.8). By Proposition 3.9(2), H admits a vector S^1-flow, say $(D_h,\boldsymbol{f}_h)$. Based on the definition of the core, it suffices to show that H has a modulo 3-orientation D_h' which can be obtained from D_h by reversing the orientations of all edges of some $E_i(H)$'s.

By Proposition 3.9(1), we have

$$|d^+_{D_h}(v)-d^-_{D_h}(v)|=\left|\sum_{i=1}^{b}\epsilon_i(v)\right|=\left|\sum_{i=1}^{b}\epsilon_i\right|.$$

Thus, if $\sum_{i=1}^{b}\epsilon_i\equiv 0 \pmod 3$, then let $D_h'=D_h$ and D_h itself is a modulo 3-orientation of H. Next we assume $\sum_{i=1}^{b}\epsilon_i\not\equiv 0 \pmod 3$.

If $k^*=3$, by Lemma 3.11(a), $\boldsymbol{\epsilon}$ has exactly three coordinates with absolute value equal to 1. Without loss of generality, we assume $|\epsilon_i|=1$ for each $i=1,2,3$ and $\epsilon_j=0$ for all j with $4\leq j\leq b$. By Proposition 3.9(4), $E_i(H)$ is a perfect matching of H for each $i=1,2,3$. Thus we can obtain a modulo 3-orientation from D_h by reversing the orientation of all edges of $E_i(H)$ for each i with $\epsilon_i=-1$.

Now we assume $k^*=5$. Then $|\epsilon_i|\leq 2$ for each i by Lemma 3.11(a). Note that $E_i(H)$ is a perfect matching if $|\epsilon_i|=1$. We first assume that $|\epsilon_i|=|\epsilon_j|=1$ for some $i\neq j$. If $\epsilon_i=\epsilon_j$, then we reverse the orientation of all edges of $E_i(H)$ and all edges of $E_t(H)$ for each $t\neq i,j$ and $\epsilon_t<0$. If $\epsilon_i=-\epsilon_j$, we reverse the orientation of all edges of $E_t(H)$ for each $t\neq i,j$ and $\epsilon_t<0$. In either case, we obtain a modulo 3-orientation D_h' of H.

Now we assume that $\boldsymbol{\epsilon}$ has at most one coordinate with absolute value 1. Without loss of generality, we assume either $\boldsymbol{\epsilon}=(1,-2,2)$ or $\boldsymbol{\epsilon}=(1,2,2)$. By Proposition 3.9(4), $E_2(H)$ and $E_3(H)$ induce two edge-disjoint spanning 2-regular subgraphs of H, and $E_1(H)$ is a perfect matching. If $\boldsymbol{\epsilon}=(1,-2,2)$, we reverse the orientation of all edges of $E_3(H)$. If $\boldsymbol{\epsilon}=(1,2,2)$, we reverse the orientation of all edges of $E_1(H)$. In either case, we obtain a modulo 3-orientation D_h' of H. ∎

LEMMA 3.13. *Suppose that a graph G admits a vector S^1-flow $(D,\boldsymbol{f})$ with rank one and $\boldsymbol{\epsilon}=\langle\epsilon_1,\epsilon_2,\ldots,\epsilon_a,\ldots,\epsilon_b\rangle$ with $|\sum_{j>a}\epsilon_i|\geq 2$ for some $1\leq a<b$. Then G has a modulo 3-orientation D' such that D' agrees with D on $\bigcup_{i=1}^{a}E_i(G)$.*

Proof. Similar to the discussion in Lemma 3.12, we only need to prove that H has a modulo 3-orientation D'_h such that D'_h agrees with D_h on $\bigcup_{i=1}^{a} E_i(H)$, where H is the core of G. We further assume that $\sum_{i=1}^{b} \epsilon_i \equiv t \pmod 3$ for some $t = 1$ or 2.

By Proposition 3.9(1) and $|\sum_{j>a} \epsilon_i| \geq 2$, H has a vertex bipartition (A, B) and has two edge-disjoint perfect matchings M_1, M_2 such that $M_1 \subseteq E_{j_1}(H), M_2 \subseteq E_{j_2}(H)$ for some $j_1, j_2 > a$ (it is possible that $j_1 = j_2$) and M_1, M_2 have the same orientation either both from A to B or both from B to A. We can obtain a modulo 3-orientation by reversing the orientations of all edges of $2t \pmod 3$ members of $\{M_1, M_2\}$ in D_h. $\square$

LEMMA 3.14. *Suppose that a graph G admits a vector S^1-flow $(D, \boldsymbol{f})$ with rank two. Suppose that $\{\boldsymbol{\eta}, \boldsymbol{\phi}\}$ are two integer-valued vectors that form a basis of $S(\boldsymbol{f})$, where*

$$\boldsymbol{\eta} = \langle 1, \eta_2, \dots, \eta_a, 0, \dots, 0\rangle, \;\; \boldsymbol{\phi} = \langle 0, \phi_2, \dots\dots, \phi_b\rangle,$$

$a < b$, and the greatest common divisor of nonzero coordinates in $\boldsymbol{\phi}$ is 1. If $V(G)$ can be partitioned into V_1 and V_2, where

$$V_1 = \{v \in V(G) : \boldsymbol{\epsilon}(v) = \pm\boldsymbol{\eta}\}, \quad V_2 = \{v \in V(G) : \boldsymbol{\epsilon}(v) = \pm\boldsymbol{\phi}\},$$

then G has a modulo 3-orientation if one of the following statements is satisfied:

(1) $\sum_{i=1}^{a} |\eta_i| \leq 5$ *and* $|\sum_{j>a}^{b} \phi_j| \geq 2$.

(2) $\sum_{i=1}^{a} \eta_i \equiv 2 \pmod 3$, *and for some* $2 \leq j \leq a$, $\eta_j = 1$ *and* $|\phi_j| \geq 2$.

Proof. Denote $\eta_1 = 1$ and $\phi_1 = 0$. We can assume that every vertex of G is unsplittable by Lemma 3.5(2). Let

$$V_{11} = \{v \in V(G) : \boldsymbol{\epsilon}(v) = \boldsymbol{\eta}\}, \; V_{12} = \{v \in V(G) : \boldsymbol{\epsilon}(v) = -\boldsymbol{\eta}\},$$
$$V_{21} = \{v \in V(G) : \boldsymbol{\epsilon}(v) = \boldsymbol{\phi}\}, \; V_{22} = \{v \in V(G) : \boldsymbol{\epsilon}(v) = -\boldsymbol{\phi}\}.$$

Then $G[V_j]$ is a bipartite graph with vertex bipartition (V_{j1}, V_{j2}) for $j = 1, 2$. Note that $E_1(G)$ is a perfect matching of $G[V_1]$. Thus $|V_{11}| = |V_{12}| = \frac{1}{2}|V_1|$. Therefore the balanced vector of the edge cut $[V_1, V_2]$ is

$$\sum_{v \in V_1} \boldsymbol{\epsilon}(v) = \sum_{v \in V_{11}} \boldsymbol{\eta} + \sum_{v \in V_{12}} -\boldsymbol{\eta} = \mathbf{0}. \tag{3}$$

Case I: $\sum_{i=1}^{a} |\eta_i| \leq 5$ and $|\sum_{j>a}^{b} \phi_j| \geq 2$.

Let G_i be the graph obtained from G by contracting V_{3-i} into a new vertex v_{3-i} for each $i = 1, 2$. Let $(D_1, \boldsymbol{f}_1)$ be the restriction of the vector S^1-flow $(D, \boldsymbol{f})$ of G to G_1. Then $(D_1, \boldsymbol{f}_1)$ is a vector S^1-flow of G_1. By (3), $\boldsymbol{\epsilon}(v_2) = \mathbf{0}$ and thus $(D_1, \boldsymbol{f}_1)$ is a vector S^1-flow with rank one. Since $\eta_1 = 1$, $\boldsymbol{\eta}$ is the elementary balanced vector of $(D_1, \boldsymbol{f}_1)$.

Since $\sum_{i=1}^{a} |\eta_i| \leq 5$, by Lemma 3.12, G_1 has a modulo 3-orientation D'_1 such that either $D'_1 = D_1$ or D'_1 can be obtained from D_1 by reversing the orientation of all edges of some $E_{i_j}(G_1)$'s. Denote $E_{i_j}(G_1)$, $j = 1, \dots, t$, the set of all edges whose orientation has been reversed. Then $i_j \leq a$.

Let D' be the orientation obtained from D by reversing the orientation of all edges of $\bigcup_{j=1}^{t} E_{i_j}(G)$. Thus $(D', \boldsymbol{f}')$ is a vector S^1-flow of G, where $\boldsymbol{f}'(e) = -\boldsymbol{f}(e)$ if $e \in \bigcup_{j=1}^{t} E_{i_j}(G)$ and $\boldsymbol{f}'(e) = \boldsymbol{f}(e)$ otherwise. For each $i \leq b$, let $\phi'_i = -\phi_i$ if the orientation of all edges of $E_i(G)$ are reversed and $\phi'_i = \phi_i$ otherwise. Since $i_j \leq a$ for each $j = 1, \dots, t$, the balanced vector of each vertex $v \in V_2$ is either $\boldsymbol{\phi}'$ or $-\boldsymbol{\phi}'$, where

$$\boldsymbol{\phi}' = \langle 0, \phi'_2, \dots, \phi'_a, \phi_{a+1}, \dots, \phi_b\rangle.$$

Similarly, G_2 admits a vector S^1-flow $(D_2, \boldsymbol{f}_2)$ which is a restriction of $(D', \boldsymbol{f}')$ with rank one. Since the greatest common divisor of nonzero coordinates of $\boldsymbol{\phi}$ is 1, $\boldsymbol{\phi}'$ is the elementary balanced vector of $(D_2, \boldsymbol{f}_2)$. Since $|\sum_{j>a}^{b} \phi'_j| = |\sum_{j>a}^{b} \phi_j| \geq 2$, by Lemma 3.13, G_2 has a modulo 3-orientation D'_2 such that D'_2 agrees with D_2 on $\bigcup_{i=1}^{a} E_i(G_2)$. Since D'_1 and D_2 are restrictions of D', D'_1 agrees with D_2 on $[V_1, V_2] \subseteq \bigcup_{i=1}^{a} E_i(G_2)$. Thus D'_1 agrees with D'_2 on $[V_1, V_2]$. Note that $D'_1 \cap D'_2 = [V_1, V_2]$. Therefore the union orientation $D'_1 \cup D'_2$ is a proper modulo 3-orientation of the graph G.

Case II: $\sum_{i=1}^{a} \eta_i \equiv 2 \pmod 3$, and for some j with $2 \leq j \leq a$, $\eta_j = 1$ and $|\phi_j| \geq 2$.

Since each vertex of G is balanced with balanced vector either $\pm\boldsymbol{\eta}$ or $\pm\boldsymbol{\phi}$, for each vertex $v \in V_1$ and each vertex $u \in V_2$, we have

$$|d_D^+(v) - d_D^-(v)| = \left|\sum_{i=1}^{a} \eta_i\right| \quad \text{and} \quad |d_D^+(u) - d_D^-(u)| = \left|\sum_{i=1}^{b} \phi_i\right|.$$

Since $\eta_1 = 1$ and $\phi_1 = 0$, $E_1(G)$ is a perfect matching in $G[V_1]$ and no vertex in V_2 is incident with an $\boldsymbol{\alpha}_1$-edge. Let D_1 be the orientation obtained from D by reversing the orientations of all $\boldsymbol{\alpha}_1$-edges of $E_1(G)$. Since $\sum_{i=1}^{a} \eta_i \equiv 2 \pmod 3$, for each vertex $v \in V_1$ and each vertex $u \in V_2$, we have

$$|d_{D_1}^+(v) - d_{D_1}^-(v)| \equiv 0 \pmod 3,\ d_{D_1}^+(u) = d_D^+(u),\ \text{and}\ d_{D_1}^-(u) = d_D^-(u).$$

If $\sum_{i=2}^{b} \phi_i \equiv 0 \pmod 3$, then D_1 is a modulo 3-orientation of G which proves the lemma.

Now we assume $\sum_{i=2}^{b} \phi_i \not\equiv 0 \pmod 3$ and regard the D_1 as digraphs. Let L be the subgraph of D_1 induced by $(E_1(G) \cup E_j(G)) \cap (E(G[V_1]) \cup E(V_1, V_2))$, the set of all $\boldsymbol{\alpha}_1$-edges and all $\boldsymbol{\alpha}_j$-edges with at least one end vertex in V_1. Since $\eta_1 = \eta_j = 1$, for each vertex $v \in V_1$, $d_L^+(v) = d_L^-(v) = 1$ and for each vertex u in $V_2 \cap V(L)$, either $d_L^+(u) = 0$ or $d_L^-(u) = 0$. Therefore, L can be decomposed into a collection of edge-disjoint directed circuits with all vertices in V_1 and directed paths with both end vertices in V_2. Note that all the directed circuit are contained in $G[V_1]$. Let $\{P_1, \ldots, P_t\}$ be the collection of directed paths in L and x_i, y_i be the head and tail of P_i, respectively, for each $i = 1, 2, \ldots, t$.

Construct a new digraph D_2 from D_1 by replacing each directed path P_i with a new edge e_i oriented from y_i to x_i and assigned $\boldsymbol{\alpha}_j$ for each $i = 1, 2, \ldots, t$. Let K denote the underlying graph of D_2 and $\boldsymbol{f}_2$ be the resulting "flow" function of K. Note that $(D_2, \boldsymbol{f}_2)$ is not a vector S^1-flow of K since all the vertices of V_1 are not balanced. But each vertex of V_2 is balanced with balanced vector either $\boldsymbol{\phi}$ or $-\boldsymbol{\phi}$ in K. By the construction of K, each $\boldsymbol{\alpha}_j$-edge of K is either in $K[V_2]$ or in the alternating $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_j)$-circuits of $K[V_1]$. Thus the subgraph induced by $E_j(K[V_2])$ is a $|\phi_j|$-regular bipartite graph with bipartition (V_{21}, V_{22}). Since $|\phi_j| \geq 2$, K has at least two edge-disjoint perfect matchings M_1, M_2 such that M_1 and M_2 have the same orientation which is either both from V_{21} to V_{22}, or both from V_{22} to V_{21}.

By the definition of D_2,

$$|d_{D_2}^+(v) - d_{D_2}^-(v)| = |d_{D_1}^+(v) - d_{D_1}^-(v)| \equiv 0 \pmod 3$$

for each $v \in V_1$, and

$$|d_{D_h}^+(2) - d_{D_2}^-(u)| = |d_{D_1}^+(u) - d_{D_1}^-(u)| \equiv r \pmod 3$$

for each $u \in V_2$, where $r = 1$ or 2. One can obtain a modulo 3-orientation D_2' of K by reversing the orientation of $2r \pmod 3$ members of $\{M_1, M_2\}$.

Now we construct D_1' from D_1 as follows: for each reversed edge e in D_2', if $e = e_i$ for some $i = 1, 2 \ldots, t$, then we reverse the orientation of all edges of P_i in D_1. Otherwise, we reverse the orientation of e in D_1. In either case, we obtain a desired modulo 3-orientation D_1' of G. ◻

3.3. Vector S^1-flows with rank two. In this subsection we will prove Theorem 1.10 for the case of vector flows with rank two.

THEOREM 3.15. *If G has a vector S^1-flow with rank two, then G admits a nowhere-zero integer 3-flow.*

Proof. Since G admits a vector S^1-flow, G is bridgeless. Let $(D, \boldsymbol{f})$ be a vector S^1-flow with rank two. By Proposition 3.3, it is equivalent to show that G admits a modulo 3-orientation. By Lemma 3.5(2), we can assume that each vertex of G is unsplittable. Note that we can further assume that G has an odd-k-edge cut with $2 < k < 7$. Otherwise, by Theorem 1.3, G admits a nowhere-zero integer 3-flow. Denote the balanced vector of the smallest odd-edge cut by

$$\boldsymbol{\eta} = \langle \eta_1, \eta_2, \ldots, \eta_b \rangle.$$

Thus $\|\boldsymbol{\eta}\| \in \{3, 5\}$. In either case, by Lemma 3.11(a), there exists some i such that $|\eta_i| = 1$. Without lose of generality, assume that $|\eta_1| = 1$ and $\boldsymbol{\eta} = \langle \eta_1, \eta_2, \ldots, \eta_a, 0, \ldots, 0 \rangle$, where $a \le b$ and $\eta_i \neq 0$ for each $1 \le i \le a$. We further assume $\eta_i \ge 0$ for each $i \le b$, otherwise we reverse the orientation of all $\boldsymbol{\alpha}_i$-edges if $\eta_i < 0$ and negate their flow values.

For each $\boldsymbol{\alpha}_1$-edge uv directed from u to v in D, we subdivide this edge into a path of length 3 by inserting two new vertices w_1 and w_2. Denote this path by uw_1w_2v. Orient w_1w_2 from w_1 to w_2 and assign w_1w_2 with the vector $\boldsymbol{\alpha}_1$. Replace w_1u with $\|\boldsymbol{\eta}\| - 1$ multiple edges oriented from w_1 to u and assign these $\|\boldsymbol{\eta}\| - 1$ edges with vectors in $\{\boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_b\}$ such that w_1 is balanced with the balanced vector $\boldsymbol{\eta}$. Similarly, replace vw_2 with $\|\boldsymbol{\eta}\| - 1$ multiple edges oriented from v to w_2 and assign these $\|\boldsymbol{\eta}\| - 1$ edges with vectors in $\{\boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_b\}$ such that w_2 is balanced with the balanced vector $-\boldsymbol{\eta}$. Denote the resulting graph by G'. Note that G' admits a vector S^1-flow, say $(D', \boldsymbol{f}')$, and has the same balanced vector space as G. Moreover, if G' has a modulo 3-orientation, then G has a modulo 3-orientation. Similarly, by Lemma 3.5(2), we can assume that each vertex in G' is unsplittable.

Now we choose a vector $\boldsymbol{\phi} = \langle \phi_1, \phi_2, \ldots, \phi_b \rangle$ of $S(\boldsymbol{f'})$ such that

(1) each ϕ_i is an integer and $\phi_1 = 0$;

(2) $\{\boldsymbol{\eta}, \boldsymbol{\phi}\}$ is a basis of $S(\boldsymbol{f'})$;

(3) subject to (1) and (2), $\|\boldsymbol{\phi}\|$ is as small as possible.

Remark. Such $\boldsymbol{\phi}$ does exist in $(D', \boldsymbol{f}')$ because there are some vertices with the first coordinate equal to zero in their balanced vectors. Moreover, all those balanced vectors are linearly independent of $\boldsymbol{\eta}$.

CLAIM 3.1. *For each vertex $v \in V(G')$, the balanced vector of v is either $\boldsymbol{\eta}$ or $-\boldsymbol{\eta}$ or is an integer multiple of $\boldsymbol{\phi}$.*

Proof. If v is incident with an $\boldsymbol{\alpha}_1$-edge, then by the definition of G', $\boldsymbol{\epsilon}(v)$ is either $\boldsymbol{\eta}$ or $-\boldsymbol{\eta}$. Otherwise, we have $\epsilon_1(v) = 0$. Since $\{\boldsymbol{\eta}, \boldsymbol{\phi}\}$ is a basis of $S(\boldsymbol{f'})$, we have $\boldsymbol{\epsilon}(v) = p\boldsymbol{\eta} + q\boldsymbol{\phi}$ for some rational numbers p and q. Since $\epsilon_1(v) = \phi_1 = 0$ and $\eta_1 = 1$, we have $p = 0$. Thus $\boldsymbol{\epsilon}(v) = q\boldsymbol{\phi}$. Note that the greatest common divisor of nonzero coordinates in $\boldsymbol{\phi}$ is 1 by the minimality of $\|\boldsymbol{\phi}\|$. Since each ϕ_i and each $\epsilon_i(v)$ are integers, q must be an integer. ◻

Note that $\eta_j = 0$ if $j \geq a+1$. So, if $\phi_j < 0$ for some $j \geq a+1$, we may reverse the orientation of all $\boldsymbol{\alpha}_j$-edges and negate the flow values to make $\phi_j > 0$ without affecting $\boldsymbol{\eta}$. Thus we can assume $\phi_j \geq 0$ for each $j \geq a+1$. By relabeling the $\boldsymbol{\alpha}_i$'s, we may further assume $\eta_2 \geq \cdots \geq \eta_a > 0$ and $\phi_{a+1} \geq \cdots \geq \phi_b \geq 0$. In summary, we assume

$$\boldsymbol{\eta} = \langle \eta_1, \ldots, \eta_a, 0, \ldots, 0 \rangle \text{ and } \boldsymbol{\phi} = \langle \phi_1, \phi_2, \ldots, \phi_b \rangle,$$

which satisfy the following:

(1) $\eta_1 = 1$, $\|\boldsymbol{\eta}\| \in \{3, 5\}$, and $\eta_2 \geq \eta_3 \geq \cdots \geq \eta_a > 0$.

(2) $\phi_1 = 0$ and $\phi_{a+1} \geq \cdots \geq \phi_b \geq 0$.

Since G' is unsplittable, by Claim 3.1 and Lemma 3.7(2), we further split each vertex v whose balanced vector is $q\boldsymbol{\phi}$ into $|q|$ vertices with balanced vector either $\boldsymbol{\phi}$ or $-\boldsymbol{\phi}$. Let H denote the graph after the splitting operation. Then H admits a vector S^1-flow by Lemma 3.5(1), denoted by $(D_h, \boldsymbol{f}_h)$, such that the balanced vector of each vertex is in $\{\pm\boldsymbol{\eta}, \pm\boldsymbol{\phi}\}$. Therefore $V(H)$ can be partitioned into V_1 and V_2, where

$$V_1 = \{v \in V(H) : \boldsymbol{\epsilon}(v) = \pm\boldsymbol{\eta}\} \text{ and } V_2 = \{v \in V(H) : \boldsymbol{\epsilon}(v) = \pm\boldsymbol{\phi}\}.$$

Note that by Lemma 3.5(2), it is sufficient to show that H has a modulo 3-orientation. Suppose to the contrary that H does not have a modulo 3-orientation.

CLAIM 3.2. $\sum_{j=a+1}^{b} \phi_j \leq 1$ *and* $a \geq 4$. *Moreover,*

(1) $\phi_{a+1} \in \{0, 1\}$ *and* $\phi_{a+2} = \cdots = \phi_b = 0$*;*

(2) $\|\boldsymbol{\eta}\| = 5$ *and either* $\boldsymbol{\eta} = \langle 1, 1, 1, 1, 1, 0, \ldots, 0 \rangle$ *or* $\boldsymbol{\eta} = \langle 1, 2, 1, 1, 0, \ldots, 0 \rangle$.

Proof. If $\sum_{j=a+1}^{b} \phi_j \geq 2$, then by Lemma 3.14(1), H has a modulo 3-orientation since $\sum_{i=1}^{a} |\eta_i| \leq 5$, a contradiction. Thus $\sum_{j=a+1}^{b} \phi_j \leq 1$. This implies that $\phi_{a+1} \in \{0, 1\}$ and $\phi_{a+2} = \cdots = \phi_b = 0$.

If $a \leq 3$, then $a = 3$ since $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are linearly independent. Since $\|\boldsymbol{\eta}\| = 3$ or 5, we have $\eta_2 = \eta_3$. Denote $t = \eta_2$. Then we have $\boldsymbol{\alpha}_1 + t\boldsymbol{\alpha}_2 + t\boldsymbol{\alpha}_3 = \mathbf{0}$. Since $\phi_1 = 0$, we have $\phi_4 \neq 0$ implying $|\phi_4| = 1$. Thus $\phi_2\boldsymbol{\alpha}_2 + \phi_3\boldsymbol{\alpha}_3 + \phi_4\boldsymbol{\alpha}_4 = \mathbf{0}$, where $|\phi_4| = 1$, a contradiction to Lemma 3.11(2). Therefore $a \geq 4$. This implies that $\|\boldsymbol{\eta}\| = 5$ and either $\boldsymbol{\eta} = \langle 1, 1, 1, 1, 1, 0, \ldots, 0 \rangle$ or $\boldsymbol{\eta} = \langle 1, 2, 1, 1, 0, \ldots, 0 \rangle$. ◻

CLAIM 3.3. $\|\boldsymbol{\phi}\| \geq 5$.

Proof. Suppose to the contrary that $\|\boldsymbol{\phi}\| < 5$. By Lemma 3.11(1), $\|\boldsymbol{\phi}\| = 3$ and $\boldsymbol{\phi}$ has exactly three coordinates with absolute value equal to 1. If there exist i, j such that $\phi_i = \phi_j = t \in \{-1, 1\}$ and $2 \leq i < j \leq a$, then it is easy to see that $(\boldsymbol{\eta} - t\boldsymbol{\phi})$ has length either 2 or 4, a contradiction to Lemma 3.11(b). Otherwise, by Claim 3.2, $\phi_{a+1} \leq 1$ implying $\phi_{a+1} = 1$ and there exist i, j such that $\phi_i = -\phi_j \in \{-1, 1\}$ and $2 \leq i < j \leq a$. By Claim 3.2(2), we have $\eta_j = 1$. We reverse the orientations of all $\boldsymbol{\alpha}_k$-edges for each $k \in \{j, a+1\}$ with $\phi_k \neq \phi_i$ and thus obtain a modulo 3-orientation of H, a contradiction. ◻

CLAIM 3.4. $\eta_j = |\phi_j| = 1$ *for some* $j \in \{2, 3, 4, 5\}$ *and* $\phi_{a+1} = 1$.

Proof. Since H does not have a modulo 3-orientation, by Lemma 3.14(2), we have that for each $j = 2, \ldots, a$, if $\eta_j = 1$, then $|\phi_j| \leq 1$. Suppose to the contrary that $|\phi_j| = 0$ for each j with $\eta_j = 1$.

By Claim 3.2(2), either $\boldsymbol{\eta} = \langle 1, 1, 1, 1, 1, 0, \ldots, 0 \rangle$ or $\boldsymbol{\eta} = \langle 1, 2, 1, 1, 0, \ldots, 0 \rangle$. Thus $\phi_3 = \cdots = \phi_a = 0$. By Claim 3.2(1), $\phi_{a+1} \leq 1$ and $\phi_j = 0$ for each $j \geq a+2$. Thus we have $\phi_2\boldsymbol{\alpha}_2 + \phi_a\boldsymbol{\alpha}_{a+1} = \mathbf{0}$. Since $\boldsymbol{\alpha}_2$ and $\boldsymbol{\alpha}_{a+1}$ are linearly independent, we have $\phi_2 = \phi_{a+1} = 0$. This implies that $\boldsymbol{\phi}$ is a zero vector, a contradiction to the choice of $\boldsymbol{\phi}$. Therefore there must be some $j \in \{2, 3, 4, 5\}$ such that $\eta_j = |\phi_j| = 1$.

By Claim 3.2(1), if $\phi_{a+1} \neq 1$, then $\phi_{a+1} = 0$ implying that $\phi_i = 0$ for each $i \geq a+1$. Since by Claim 3.3, $\sum_{i=1}^{a} |\phi_i| \geq 5 = \sum_{i=1}^{a} \eta_i$ and $\phi_1 = 0$, there must exist some $j \in \{2, \ldots, a\}$ such that $1 \leq \eta_j < |\phi_j|$. Since $\eta_i = 1$ implies $|\phi_i| \leq 1$, we have $\eta_j \geq 2$ and $|\phi_j| \geq 3$. By Claim 3.2, $\boldsymbol{\eta} = \langle 1, 2, 1, 1, 0, \ldots, 0\rangle$ and so $j = 2$. Thus we have

$$\phi_2\boldsymbol{\alpha}_2 + \phi_3\boldsymbol{\alpha}_3 + \phi_4\boldsymbol{\alpha}_4 = \mathbf{0},$$

where $|\phi_2| \geq 3$ and $|\phi_3|, |\phi_4| \leq 1$, a contradiction to Lemma 3.11(a). This proves $\phi_{a+1} = 1$. ∎

The final step.

If $\sum_{i=2}^{b} \phi_i \equiv 0 \pmod 3$, then one can obtain a modulo 3-orientation of H by reversing the orientation of all edges of $E_1(H)$ in D_h since $\phi_1 = 0$, a contradiction.

If $\sum_{i=2}^{b} \phi_i \equiv 2 \pmod 3$, then one can obtain a modulo 3-orientation of H by reversing the orientation of all edges of $E_1(H) \cup E_{a+1}(H)$ in D_h since $\phi_{a+1} = \eta_1 = 1$ and $\eta_{a+1} = \phi_1 = 0$, a contradiction again. Therefore $\sum_{i=2}^{b} \phi_i \equiv 1 \pmod 3$.

By Claim 3.4, $\eta_j = |\phi_j| = 1$ for some $j \leq a$. Note that $\phi_{a+1} = 1$ and $\eta_{a+1} = 0$. If $\phi_j = 1$, reverse the orientation of all edges in $E_j(H) \cup E_{a+1}(H)$ in D_h. Otherwise, reverse the orientation of all edges of $E_j(H)$ in D_h. In either case, the resulting orientation is a modulo 3-orientation of H, a contradiction. This completes the proof of Theorem 3.15. ∎

4. Proof of Theorem 1.12.

Proof. We only need to prove the sufficiency by Theorem 1.6. Suppose that G admits a vector S^1-flow $(D, \boldsymbol{f})$. Let v be a degree 3 vertex of G. Without loss of generality, we assume $E^-(v) = \varnothing$. Denote $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\} = \{\boldsymbol{f}(e) | e \in E^+(v)\}$. Then

$$\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3 = \mathbf{0}. \tag{4}$$

We further assume that no edges of G have flow value $-\boldsymbol{\alpha}_i$ for each $i = 1, 2, 3$ since otherwise we can reverse the orientation of all $(-\boldsymbol{\alpha}_i)$-edges and negate their flow values. By (4), we have the following claim.

CLAIM 4.1. *Let $\boldsymbol{\beta} = t_1\boldsymbol{\alpha} + t_2\boldsymbol{\alpha}_2 + t_3\boldsymbol{\alpha}_2$, where t_1, t_2 and t_3 are integers. Then the following two statements are true.*

(1) *If $\boldsymbol{\beta} = \mathbf{0}$, then $t_1 = t_2 = t_3$.*

(2) *If $|\boldsymbol{\beta}| = 1$, then $\boldsymbol{\beta} \in \{\pm\boldsymbol{\alpha}_1, \pm\boldsymbol{\alpha}_2, \pm\boldsymbol{\alpha}_3\}$.*

Proof. Without loss of generality, we assume $t_1 \leq t_2 \leq t_3$. By eliminating $\boldsymbol{\alpha}_1$ using (4), we have $\boldsymbol{\beta} = (t_2 - t_1)\boldsymbol{\alpha}_2 + (t_3 - t_1)\boldsymbol{\alpha}_3$

(1) If $\boldsymbol{\beta} = \mathbf{0}$, then we have $(t_2 - t_1)\boldsymbol{\alpha}_2 + (t_3 - t_1)\boldsymbol{\alpha}_3 = \mathbf{0}$. Since $\boldsymbol{\alpha}_2$ and $\boldsymbol{\alpha}_3$ are linearly independent, $t_2 - t_1 = t_3 - t_1 = 0$ implying $t_1 = t_2 = t_3$.

(2) Assume $|\boldsymbol{\beta}| = 1$. Suppose to the contrary $\boldsymbol{\beta} \notin \{\pm\boldsymbol{\alpha}_1, \pm\boldsymbol{\alpha}_2, \pm\boldsymbol{\alpha}_3\}$. Then $\boldsymbol{\beta}$ is linearly independent of $\boldsymbol{\alpha}_2$ and $\boldsymbol{\alpha}_3$. Thus $t_2 - t_1 \neq 0$ and $t_3 - t_1 \neq 0$. By Lemma 3.11(a), we have $|t_2 - t_1| = |t_3 - t_1|$. Since $t_1 \leq t_2 \leq t_3$, we have $t_2 - t_1 = t_3 - t_1$. Thus $\boldsymbol{\beta} = (t_2 - t_1)(\boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3) = -(t_2 - t_1)\boldsymbol{\alpha}_1$. Since $|\boldsymbol{\beta}| = |\boldsymbol{\alpha}_1| = 1$, we have $t_2 - t_1 = |\boldsymbol{\beta}| = 1$ and $\boldsymbol{\beta} = -\boldsymbol{\alpha}_1$. This contradiction implies $\boldsymbol{\beta} \in \{\pm\boldsymbol{\alpha}_1, \pm\boldsymbol{\alpha}_2, \pm\boldsymbol{\alpha}_3\}$. ∎

CLAIM 4.2. *For each degree 3 vertex u, $\{\boldsymbol{f}(e) | e \in E(u)\} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}$.*

Proof. Since $G[V_3]$ is connected, it suffices to show that for each degree 3 vertex u adjacent to v, $\{\boldsymbol{f}(e) | e \in E(u)\} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}$. Without loss of generality, assume $\boldsymbol{f}(vu) = \boldsymbol{\alpha}_3$. Since $d(u) = 3$, there are two other vectors $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in S^1$ such that

$$\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2 - \boldsymbol{\alpha}_3 = \mathbf{0}. \tag{5}$$

Adding (4) to (5), we have

$$\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2 = \mathbf{0}.$$

Since all vectors in S^1 are unit vectors, the vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_1,$ and $\boldsymbol{\beta}_2$ form a parallelogram in $\mathbb{R}^2$. Thus $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2\} = \{-\boldsymbol{\alpha}_1, -\boldsymbol{\alpha}_2\}$, so we have $E^+(u) = \varnothing$ since there is no edge e in G having $\boldsymbol{f}(e) = -\boldsymbol{\alpha}_i$ for each $i = 1, 2, 3$. Therefore, $\{\boldsymbol{f}(e) | e \in E(u)\} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}$. ∎

Claim 4.3. *For each $e \in E(G)$, $\boldsymbol{f}(e) \in \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}$.*

Proof. Let $E' = \{e \in E(G) | \boldsymbol{f}(e) \neq \boldsymbol{\alpha}_i, \text{for each } i = 1, 2, 3\}$ and $G' = G[E']$. We only need to show $E' = \varnothing$. Suppose to the contrary that $E' \neq \varnothing$.

By Claim 4.2, G' is a subgraph of $G - V_3$. Thus it is acyclic. If $E' \neq \varnothing$, then there exists a degree 1 vertex w in G' since G' is acyclic. Let wz be the edge in E' with flow value $\boldsymbol{\beta}$. Then except for wz, the flow value of each other edge incident with w is $\boldsymbol{\alpha}_i$ for some $i \in \{1, 2, 3\}$. Without lose of generality, we assume that wz is oriented from w to z. Then the balanced vector of w is

$$\boldsymbol{\epsilon}(w) = \boldsymbol{\beta} + r_1\boldsymbol{\alpha}_1 + r_2\boldsymbol{\alpha}_2 + r_3\boldsymbol{\alpha}_3 = \mathbf{0},$$

where r_1, r_2, and r_3 are integers. Thus $\boldsymbol{\beta} = -r_1\boldsymbol{\alpha}_1 - r_2\boldsymbol{\alpha}_2 - r_3\boldsymbol{\alpha}_3$. By Claim 4.1, $\boldsymbol{\beta} \in \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}$ because $|\boldsymbol{\beta}| = 1$ and no edge has flow value $-\boldsymbol{\alpha}_i$ for each $i = 1, \ldots, k$. This contradicts the assumption that $\boldsymbol{\beta} \notin \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}$. Therefore, $E' = \varnothing$. ∎

Claim 4.3 implies that the balanced vector of each vertex v is $\boldsymbol{\epsilon}(v) = t_1\boldsymbol{\alpha}_1 + t_2\boldsymbol{\alpha}_2 + t_3\boldsymbol{\alpha}_3 = \mathbf{0}$ for some integers t_1, t_2, t_3. By Claim 4.1, $t_1 = t_2 = t_3$ and thus $\boldsymbol{\epsilon}(v) = t_1(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3)$. This implies that the rank of $S(\boldsymbol{f})$ is one. Therefore G admits a nowhere-zero integer 3-flow by Theorem 1.10. ∎

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