SIAM J. DISCRETE MATH.
Vol. 22, No. 2, pp. 500–519

REALIZING DEGREE SEQUENCES WITH GRAPHS HAVING NOWHERE-ZERO 3-FLOWS*

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Abstract. The following open problem was proposed by Archdeacon: Characterize all graphical sequences π such that some realization of π admits a nowhere-zero 3-flow. The purpose of this paper is to resolve this problem and present a complete characterization: A graphical sequence $\pi = (d_1, d_2, \ldots, d_n)$ with minimum degree at least two has a realization that admits a nowhere-zero 3-flow if and only if $\pi \neq (3^4, 2)$, $(k, 3^k)$, $(k^2, 3^{k-1})$, where k is an odd integer.

Key words. degree sequence, graph, integer flow, characterization

AMS subject classifications. 05C70, 05C38, 05C45

DOI. 10.1137/070687372

1. Introduction. Let $G = (V, E)$ be a graph and let k be a positive integer. An ordered pair (D, ϕ) is called a *k-flow* of G if $D = (V, A)$ is an orientation of G and $\phi : A \mapsto Z_k$ is an assignment of flows such that, for every vertex v,

$$\sum_{e \in E^+(v)} \phi(e) \equiv \sum_{e \in E^-(v)} \phi(e) \pmod{k},$$

where Z_k is the set Z/kZ of integers modulo k, and $E^+(v)$ (resp., $E^-(v)$) is the set of all arcs in A with tail v (resp., head v). We say that (D, ϕ) is a *nowhere-zero flow* if $\phi(e) \neq 0$ for any $e \in A$. This concept was introduced by Tutte [19], and the theory of nowhere-zero flows provides an interesting way to generalize theorems about region-coloring planar graphs to general graphs; major open problems in this area are Tutte's celebrated 3-, 4-, and 5-flow conjectures. Interested readers are referred to Jaeger [8] and Seymour [17] for the main ideas of this subject and to Tutte [20] and Zhang [21] for in-depth accounts.

An integer-valued sequence $\pi = (d_1, d_2, \ldots, d_n)$ is called *graphical* if there is a *simple* graph G so that the degree sequence of G is exactly the same as π; such a graph G is called a *realization* of π. For simplicity, we shall also write a graphical sequence in terms of multiplicities, for instance, $(6, 4, 4, 3, 3, 3, 3) = (6, 4^2, 3^4)$. The problem of realizing degree sequences with graphs enjoying certain properties has been a subject of extensive research. Recently a surprising application of graph realization with 4-flows has been found in the design of critical partial Latin squares [5, 15], which leads to the proof [14] of the so-called *simultaneous edge-coloring conjecture*

*Received by the editors April 4, 2007; accepted for publication (in revised form) November 28, 2007; published electronically March 20, 2008.
http://www.siam.org/journals/sidma/22-2/68737.html
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by Keedwell [10, 11] and Cameron [2]. In this paper we study a closely related open problem proposed by Archdeacon.

PROBLEM 1.1 (see [1]). *Characterize all graphical sequences π such that some realization of π admits a nowhere-zero 3-flow.*

Our objective is to resolve this problem and to present a complete characterization.

THEOREM 1.2. *A graphical sequence $\pi = (d_1, d_2, \ldots, d_n)$ with minimum degree at least two has a realization that admits a nowhere-zero 3-flow if and only if $\pi \neq (3^4, 2)$, $(k, 3^k)$, $(k^2, 3^{k-1})$, where k is an odd integer.*

It is worthwhile pointing out that the most striking difference between [14] and the present paper is not the flow number but the proof technique. We say that a graph H is a *k-flow contractible configuration* if for every graph G containing H as a subgraph, G admits a nowhere-zero k-flow if and only if so does G/H. Actually flow contractible configurations play important roles in both papers: For the 4-flow problem, every circuit of length at most four (cf. Seymour [16] and Catlin [3]) is contractible. The situation, however, becomes much more complicated for the 3-flow problem, and the digon is the only circuit that is 3-flow contractible [16]. So we have to appeal to other 3-flow contractible configurations. To be more precise, a graph $G = (V, E)$ is *Z_k-connected* [9] if for every $b : V \mapsto Z_k$ with $\sum_{v \in V} b(v) \equiv 0 \pmod{k}$, there exist an orientation $D = (V, A)$ of G and an assignment $\phi : A \mapsto \{1, 2, \ldots, k\}$ such that for every vertex v,

$$\sum_{e \in E^+(v)} \phi(e) - \sum_{e \in E^-(v)} \phi(e) \equiv b(v) \pmod{k}.$$

As shown in [4, 13], Z_3-connected graphs contain even wheels, triangularly connected graphs with a family of well-described exceptions, etc.; it is Z_3-connected graphs that will serve as contractible configurations in our proof.

The remainder of this paper is organized as follows. In section 2, we exhibit some basic properties concerning graphical sequences and Z_3-connectivity. In section 3, we describe some graphical sequences with Z_3-connected realizations. In section 4, we characterize certain graphical sequences π such that some realization of π admits a nowhere-zero 3-flow and contains nontrivial Z_3-connected subgraphs. In section 5, we present a proof of the main theorem (Theorem 1.2), which fully characterizes all graphical sequences that can be realized to admit nowhere-zero 3-flows.

We remark that a complete characterization of graphic sequences with Z_3-connected realizations remains an interesting problem for further study.

2. Preliminaries. Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence with $d_1 \geq d_2 \geq \cdots \geq d_n$. Throughout we reserve the symbol $\bar{\pi}$ for the sequence $(d_1 - 1, d_2 - 1, \ldots, d_{d_n} - 1, d_{d_n+1}, \ldots, d_{n-1})$, which is called the *residual sequence* obtained from π by laying off d_n. We shall frequently use the following well-known results in our proof.

LEMMA 2.1. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a sequence. Then*

(a) $\sum_{i=1}^{n} d_i$ *is even if π is graphical;*

(b) *(Hakimi* [6, 7]*; Kleitman and Wang* [12]*) π is graphical if and only if so is $\bar{\pi}$.*

LEMMA 2.2 (Tutte [18]). *A cubic graph admits a nowhere-zero 3-flow if and only if it is bipartite.*

Lemma 2.2 can be further generalized in the following way.

LEMMA 2.3. *If a graph G admits a nowhere-zero 3-flow, then the subgraph of G induced by all degree-three vertices is bipartite.*

If H is a connected subgraph of a graph G, then G *contracted by* H, denoted by G/H, is the graph obtained from G by deleting all edges in H and then identifying $V(H)$ into a single vertex. The following simple observations follow instantly from the definition of Z_k-connectivity.

LEMMA 2.4 (Jaeger [8]; Seymour [16]). *Every circuit of length at most* $k-1$ *is* Z_k*-connected.*

LEMMA 2.5 (DeVos, Xu, and Yu [4]; Lai, Xu, and Zhang [13]). *Let* H *be a* Z_3*-connected subgraph of a graph* G.

(a) *If* G/H *admits a nowhere-zero* 3*-flow, then so does* G.

(b) *If* G/H *is* Z_3*-connected, then so is* G.

In our proof these facts enable us to work on a reduced graph after a series of contractions of Z_3-connected subgraphs. The reduced graphs/graphical sequences often enjoy much nicer properties than the original ones, and therefore are much easier to manipulate.

The following two lemmas are immediate corollaries of Lemmas 2.5 and 2.4.

LEMMA 2.6. *Let* G *be a* Z_3*-connected graph, and let* G' *be obtained from* G *by adding a new vertex* v *and making it adjacent to at least two vertices of* G*. Then* G' *is* Z_3*-connected.*

LEMMA 2.7. *Let* $G=(V,E)$ *be a* Z_3*-connected graph. Then for any* $u,v\in V$ *with* $uv\notin E$*, the graph obtained from* G *by adding an edge* uv *is* Z_3*-connected.*

Let $G=(V,E)$ be a graph, and let u,v,w be three vertices of G with $uv,uw\in E$. In this paper we shall use $G_{[uv,uw]}$ to stand for the graph $G\cup\{vw\}\setminus\{uv,uw\}$.

LEMMA 2.8. *Let* $G=(V,E)$ *be a graph, and let* u,v,w *be three vertices of* G *with degree* $d(u)\geq 4$ *and* $uv,uw\in E$*. If* $G_{[uv,uw]}$ *is* Z_3*-connected, then so is* G.

A graph $G=(V,E)$ is *triangularly connected* if for every $e,f\in E$ there exists a sequence of circuits $C_1,C_2,\ldots,C_k$ such that $e\in E(C_1)$, $f\in E(C_k)$, and $|E(C_i)|\leq 3$ for $1\leq i\leq k$, and $E(C_j)\cap E(C_{j+1})\neq\emptyset$ for $1\leq j\leq k-1$. A *wheel* W_k is the graph obtained from a k-circuit C by adding a vertex v and then making it adjacent to all vertices on C. By convention, we call v the *hub* and C the *rim* of W_k. We also call W_k *odd* if k is odd and *even* otherwise.

The following lemma gives sufficient conditions for a graph to be Z_3-connected.

LEMMA 2.9 (DeVos, Xu, and Yu [4]). *Let* G *be a triangularly connected graph. Then* G *is* Z_3*-connected, provided that one of the following conditions is satisfied:*

(a) G *contains a nontrivial* Z_3*-connected subgraph;*

(b) *the minimum degree of* G *is at least four;*

(c) G *is an even wheel* W_k*, with* $k\geq 4$.

The following lemma establishes the "only if" part of our main theorem (Theorem 1.2).

LEMMA 2.10. *Let* k *be an odd integer. Then no realization of the graphical sequences* $(3^4,2)$*,* $(k,3^k)$*,* $(k^2,3^{k-1})$ *admits a nowhere-zero* 3*-flow.*

Proof. Observe the following:

• the only realization G_1 of $(3^4,2)$ is the graph obtained from K_4 (the complete graph with four vertices) by subdividing an edge precisely once;

• a realization G_2 of $(k,3^k)$ is *either* an odd wheel *or* several wheels sharing the hub, and at least one of these wheels is odd;

• the only realization G_3 of $(k^2,3^{k-1})$ is $\frac{k-1}{2}$ copies of K_4 sharing a common edge. (To justify this, let u and v be the two vertices of maximum degree. Then the subgraph obtained from G_3 by deleting u and v is 1-regular.)

By Lemma 2.3, neither G_1 nor G_2 admits a nowhere-zero 3-flow. To prove the statement concerning sequence $(k^2,3^{k-1})$, let Q_i be a copy of K_4 with vertex set $\{w_i, x_i, y_i, z_i\}$ for $i = 1, 2, \ldots, \frac{k-1}{2}$. As described above, G_3 is obtained from $Q_1, Q_2, \ldots, Q_{\frac{k-1}{2}}$ by first identifying all y_i as a single vertex y and all z_i as a single vertex z, and then replacing all edges $y_i z_i$ with a single edge yz. Assume to the contrary that G_3 admits a nowhere-zero 3-flow (D, f). Then there must exist a 3-flow (D, f_i) of each Q_i such that

• $f(e) = f_i(e)$ for every edge $e \in Q_i - \{yz\}$ (that is, $\text{supp}(f_i) \supseteq E(Q_i) - \{yz\}$) and

• $f = \sum_{i=1}^{\frac{k-1}{2}} f_i$, which implies the existence of a subscript i such that f_i is nowhere-zero on Q_i $(= K_4)$, contradicting Lemma 2.2. □

3. Z_3-connected realizations. The purpose of this section is to establish the following two theorems, which give some sufficient conditions for Z_3-connected realizations.

THEOREM 3.1. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence with $d_1 \geq d_2 \geq \cdots \geq d_n$. If $d_n \geq 3$ and $d_{n-3} \geq 4$, then π has a Z_3-connected realization.*

THEOREM 3.2. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence with $n-1 = d_1 \geq d_2 \geq \cdots \geq d_n \geq 3$. Then π has a Z_3-connected realization if and only if $\pi \neq (k, 3^k)$, $(k^2, 3^{k-1})$, where k is odd.*

Let us establish a weaker version of Theorem 3.1 before presenting a proof.

LEMMA 3.3. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence with $d_1 \geq d_2 \geq \cdots \geq d_n$. If $d_n \geq 3$ and $d_{n-2} \geq 4$, then π has a Z_3-connected realization.*

Proof. Suppose the contrary: $\pi = (d_1, d_2, \ldots, d_n)$ is a counterexample with smallest n. According to the configuration of π, we propose to consider four cases and construct a Z_3-connected realization of π in each case, thereby reaching a contradiction. Notice that $n \geq 5$ as $n - 1 \geq d_1 \geq d_{n-2} \geq 4$.

The lemma is to be proved step by step with the following observations and claims.

(1) $d_1 \geq 5$. Assume the contrary: $d_1 = 4$. Let us consider the following two cases.

Case 1. $d_1 = 4$ and $d_n = 4$. In this case $\pi = (4^n)$, the construction goes as follows: Let C be a circuit with n vertices $v_1, v_2, \ldots, v_n$, and let $G^1 = C \cup \bigcup_{i=1}^{n} \{v_i v_{i+2}\}$, where $v_{n+1} = v_1$ and $v_{n+2} = v_2$. Clearly, G^1 is 4-regular and is triangularly connected. By Lemma 2.9(b), G^1 is Z_3-connected.

Case 2. $d_1 = 4$ and $d_n = 3$. Since $d_1 = d_{n-2} = 4$ and $d_n = 3$, by Lemma 2.1(a) we have $\pi = (4^{n-2}, 3^2)$. Let G^1 be the graph constructed in Case 1, and let $G^2 = G^1 \setminus \{v_2 v_n\}$. Clearly G^2 is a realization of π. It remains to show that G^2 is Z_3-connected.

From the construction of G^1 and G^2, we see that $G^2_{[v_1 v_2, v_1 v_3]}$ is triangularly connected and contains a 2-circuit $v_2 v_3 v_2$. Since any 2-circuit is Z_3-connected (by Lemma 2.4), Lemma 2.9(a) implies that $G^2_{[v_0 v_1, v_0 v_2]}$ is Z_3-connected, and hence so is G^2 by Lemma 2.8. Therefore (1) holds.

(2) $d_2 = 4$. Suppose to the contrary that $d_2 \geq 5$. By (1), we have $d_1 \geq 5$ and $d_2 \geq 5$.

Since $n \geq 5$ and $d_{n-2} \geq 4$, we get $d_3 \geq 4$. Hence it can be seen that the residual sequence $\bar{\pi} = (\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_{n-1})$, with $\bar{d}_1 \geq \bar{d}_2 \geq \cdots \geq \bar{d}_{n-1}$, satisfies $\bar{d}_{n-1} \geq 3$ and $\bar{d}_{n-3} \geq 4$. Thus Lemma 2.1(b) and the assumption on π guarantee the existence of a Z_3-connected realization $\bar{G}$ of $\bar{\pi}$. We can then get a realization G^3 of π from $\bar{G}$ by adding a new vertex v and d_n edges joining v to the corresponding vertices in $\bar{G}$. By Lemma 2.6, G^3 is Z_3-connected. This proves (2).

We claim that (3) $d_n = 3$. Otherwise, $d_n = 4$. By (1) and (2), we have $d_1 \geq 5$ and $d_2 = 4$. So $\pi = (d_1, 4, \ldots, 4)$. By Lemma 2.1(a), d_1 is even. Thus $d_1 \geq 6$.

Set $k = \frac{d_1-4}{2}$. Let G^1 be the 4-regular triangularly connected graph exhibited in Case 1 of (1). For each $1 \le i \le k$, we subdivide the edge $v_{2i+2}v_{2i+3}$ once by a degree-two vertex u_i. Since $n-1 \ge d_1 = 2k+4$, we have $2k+3 \le n-2$. Now let us identify all u_i with v_1. Then the resulting graph G^4 is simple and is clearly a realization of π. To show that G^4 is Z_3-connected, we replace the path $v_{2i+2}v_1v_{2i+3}$ with an edge $v_{2i+2}v_{2i+3}$ for all $1 \le i \le k$; the resulting graph is precisely G^1. Since G^1 is Z_3-connected, repeated applications of Lemma 2.8 imply that so is G^4. Thus (3) follows.

By (1), (2), and (3), we have $d_1 \ge 5$, $d_2 = 4$, and $d_n = 3$. So π is either $(d_1, 4, \dots, 4, 3, 3)$ or $(d_1, 4, \dots, 4, 3)$.

If $d_{n-1} = 4$, then the residual sequence $\bar{\pi}$ satisfies the conditions of the theorem, so it admits a Z_3-connected realization $\bar{G}$, and hence so does G. It remains to consider the case when $d_{n-1} = 3$. Since $d_2 = d_3 = \cdots = d_{n-2} = 4$ and $d_{n-1} = d_n = 3$, we see that d_1 is even. So $d_1 \ge 6$. Set $k = \frac{d_1-4}{2}$. Let G^2 be the triangularly connected graph constructed in Case 2 of (1). For each $1 \le i \le k$, we subdivide the edge $v_{2i+2}v_{2i+3}$ once by a degree-two vertex u_i. Since $n-1 \ge d_n = 2k+4$, we have $2k+3 \le n-2$. Now let us identify u_i and v_1 for each $1 \le i \le k$. Then the resulting graph G^5 is simple and is clearly a realization of π. To show that G^5 is Z_3-connected, we replace the path $v_{2i+2}v_1v_{2i+3}$ with an edge $v_{2i+2}v_{2i+3}$ for all $1 \le i \le k$; then the resulting graph is precisely G^2. Since G^2 is Z_3-connected, by Lemma 2.8 so is G^5, completing the proof of the lemma. ∎

Proof of Theorem 3.1. Suppose the contrary: $\pi = (d_1, d_2, \dots, d_n)$ is a counterexample with smallest n. Notice that $n \ge 5$ as $n-1 \ge d_1 \ge d_{n-3} \ge 4$. By Lemma 3.3, we have

(1) $d_{n-2} = 3$.

Let us further make some simple observations.

(2) $d_2 = 4$. Otherwise, $d_2 \ge 5$ for $d_2 \ge d_{n-3} \ge 4$. So the residual sequence $\bar{\pi}$ satisfies the conditions of the theorem, and hence the assumption on π guarantees the existence of a Z_3-connected realization $\bar{G}$ of $\bar{\pi}$. By Lemma 2.6, we can get a Z_3-connected realization G of π from $\bar{G}$ by adding a new vertex v and d_n edges joining v to the corresponding vertices in $\bar{G}$. This contradiction implies (2).

Combining (1), (2), and the hypothesis of the theorem, we get

(3) $\pi = (d_1, 4^{n-4}, 3^3)$. So d_1 is odd by Lemma 2.1(a) and hence at least 5.

(4) $n \ge 8$. Suppose to the contrary that $n \le 7$. Since $d_1 \ge 5$ and is odd by (3), we have $d_1 = 5$ and $n \ge 6$. So $\pi = (5, 4^{n-4}, 3^3)$.

For $n = 6$, let G be the graph obtained from the complete bipartite graph $K_{2,3}$ by adding a new vertex and then making it adjacent to each vertex in the $K_{2,3}$. Clearly each edge of G is contained in a wheel W_4. Since W_4 is Z_3-connected by Lemma 2.9(c), so is G by Lemma 2.5(b).

For $n = 7$, let G be the graph obtained from W_4 by adding two adjacent vertices v_1 and v_2 and then making v_1 adjacent to the hub of W_4 and a rim vertex and making v_2 adjacent to two other rim vertices. Since both W_4 and the graph obtained from G by contracting W_4 (which results in a triangle with parallel edges) are Z_3-connected, so is G by Lemma 2.5(b); this contradiction establishes (4).

Let us distinguish between two cases according to the value of d_1.

Case 1. $d_1 \ge n-3$. Set $k = \frac{d_1-5}{2} \ge 0$ and $m = \lfloor \frac{n-4}{2} \rfloor$. Take a wheel W_{n-4} with hub w and rim $u_1u_2u_3 \dots u_{n-4}u_1$. Let H be the graph obtained from this wheel by adding k edges u_iu_{i+m} for $i = 1, 2, \dots, k$. Then the degree sequence of H is $(n-4, 4^{2k}, 3^{n-4-2k}) = (n-4, 4^{d_1-5}, 3^{n-d_1+1})$. To get a graph G with n vertices and degree sequence $\pi = (d_1, 4^{n-4}, 3^3)$, we need to add three vertices and

$[d_1 + 4(n-4) + 9 - (n-4) - 4(d_1 - 5) - 3(n - d_1 + 1)]/2 = 7$ edges to H. The construction of G goes as follows: We first add a path $P = v_1v_2v_3$ to H, then we connect w and $d_1 - (n-4)$ vertices on P, and finally add precisely one edge between each of $n - d_1 + 1$ degree-three vertices on H and P, so that there are precisely two edges between each of v_1 and v_3 and H, and there is precisely one edge between v_2 and H.

By (4), $n \geq 8$. Note that if $n = 8$, then $H = W_4$; if $n \geq 9$, then at least one edge is added to W_{n-4}, which implies that H contains an even wheel. So, by Lemma 2.9(c), H contains a Z_3-connected subgraph (an even wheel) in either case. Clearly, G/H is triangularly connected and contains a 2-circuit. Since a 2-circuit is Z_3-connected, so is G/H by Lemma 2.9(a). It follows from Lemma 2.5(b) that G is Z_3-connected, a contradiction.

Case 2. $d_1 \leq n - 4$. By (3), $d_1 \geq 5$. So in this case $n \geq d_1 + 4 \geq 9$. Let us consider the sequence $\sigma = (d_1 - 1, 4^{n-7}, 3^2)$. From the construction of G^2 and G^5 of the proof of Lemma 3.3, we deduce that σ has a Z_3-connected realization H; let u_1, u_2, u_3 denote the vertices of H with degree three and degree $d_1 - 1$, respectively. Let G be the graph obtained from H by first adding a complete graph with four vertices v_1, v_2, v_3, v_4, then deleting edge v_1v_3, and finally adding a matching of size three between $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$. Clearly, G is a realization of π. Note that G/H is a wheel W_4 with hub v_2, so by Lemma 2.9(c) it is Z_3-connected, and hence so is G by Lemma 2.5(b). This contradiction completes the proof of the theorem. □

The proof of Theorem 3.2 is based on the following lemma.

LEMMA 3.4. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 2$. Then π has a connected realization G that contains an even circuit if and only if $\pi \neq (2^n)$, $(n-1, 2^{n-1})$, where n is odd.*

Proof. It is easy to see that when n is odd

- the unique connected realization of $\pi = (2^n)$ is an odd circuit;
- the unique connected realization of $\pi = (n-1, 2^{n-1})$ is $\frac{n-1}{2}$ triangles sharing a common vertex.

Clearly neither of these two graphs contains an even circuit, so the "only if" part is established.

Let us proceed to the "if" part. Assume that $\pi \neq (2^n)$, $(n-1, 2^{n-1})$, where n is odd, but no connected realization of π contains an even circuit. We further assume that π is chosen with minimum n. Let us make some simple observations.

(1) $n \geq 5$. By the assumption on π, we have $n \neq 3$. Hence $n \geq 4$. If $n = 4$, then $\pi = (2^4)$ or $(3^2, 2^2)$ or (3^4). In each case π has a connected realization that contains an even circuit. This contradiction yields (1).

(2) $d_n = 2$. Otherwise, $d_n \geq 3$. By (1), the residual sequence $\bar{\pi}$ satisfies the condition of the lemma. Hence it admits a connected realization H that contains an even circuit by the assumption on π. We can then get a desired realization G of π from H by adding a new vertex and making it adjacent to corresponding vertices in H, a contradiction. So (2) holds.

(3) $d_2 \geq 3$. Otherwise, by (2) we have $d_2 = 2$, and so $\pi = (d_1, 2^{n-1})$, where $d_1 = 2k$ for some integer $k \geq 1$. Let H be the graph obtained from k disjoint triangles and then gluing them at a common vertex v. Note that the number of vertices in H is $2k + 1$. According to the assumption on π, we have $n \neq 2k + 1$ and $k \geq 2$. So $2k + 1 \leq n - 1$. Let G be the graph obtained from H by inserting a degree-two vertex into the edge not containing v in the first triangle, and inserting the remaining degree-two vertices (if any) into the edge not containing v in the second triangle. Then G is a connected realization of π that contains an even circuit (of length four), a contradiction. So (3) is proved.

Consider the residual sequence $\bar{\pi}$ of π. By (3), we have $d_1 - 1 \geq d_2 - 1 \geq 2$. If $\bar{\pi}$ satisfies the conditions of the lemma, then the assumption on π guarantees a connected realization H of $\bar{\pi}$ that contains an even circuit. From H we can obviously get a desired realization of π. This contradiction implies that $\bar{\pi} = (2^{n-1})$ or $(n-2, 2^{n-2})$, where $n-1$ is odd and thus n is even.

If $\bar{\pi} = (2^{n-1})$, then $\pi = (3^2, 2^{n-2})$ by (2). We can get a desired realization of π from an n-circuit by adding a chord.

If $\bar{\pi} = (n-2, 2^{n-2})$, then the unique connected realization H of $\bar{\pi}$ is $\frac{n-2}{2}$ triangles sharing a common vertex v. Note that $\pi = (n-1, 3, 2^{n-2})$ by (2). Let G be the graph obtained from H by adding a new vertex and making it adjacent to v and one other vertex. Clearly, G is a connected realization of π and contains an even circuit. This contradiction completes the proof. ∎

Proof of Theorem 3.2. The "only if" part follows instantly from Lemma 2.10. It remains to show the "if" part.

Consider the sequence $\sigma = (d_2 - 1, d_3 - 1, \ldots, d_n - 1)$. Note that $\sigma \neq (2^{n-1})$, $(n-2, 2^{n-2})$, where $n-1$ is odd, for otherwise $\pi = (n-1, 3^{n-1})$ or $((n-1)^2, 3^{n-2})$, contradicting the hypothesis on π. By Lemma 3.4, σ has a connected realization H that contains an even circuit. Let G be the graph obtained from H by adding a new vertex and making it adjacent to each vertex of H. Clearly G is a realization of π. Since G is triangularly connected and contains an even wheel, from Lemma 2.9 we deduce that G is Z_3-connected. ∎

4. Partially Z_3-connected realizations. We propose to establish the following two theorems in this section.

THEOREM 4.1. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 3$ and $d_3 \geq 5$. Then π has a realization G such that*

(a) *G admits a nowhere-zero 3-flow; and*

(b) *G has a Z_3-connected subgraph H that contains all vertices of G with degree at least four.*

THEOREM 4.2. *Let $\pi = (d_1, d_2, 4^{n-k-2}, 3^k)$ be a graphical sequence with $n-2 \geq d_1 \geq d_2 \geq 4$, $d_1 + d_2 \geq 11$, $n-3 \geq k \geq 4$, and $n \geq 9$. Then π has a realization G such that*

(a) *G admits a nowhere-zero 3-flow; and*

(b) *G has a Z_3-connected subgraph H that contains all vertices of G with degree at least four.*

Let us introduce three operations before proving these theorems, which will be used frequently in our proofs.

Let H_1 and H_2 be two disjoint graphs. A graph G *obtained by adding H_2 onto H_1 via Operation* A, B, *or* C is defined below.

Operation A. Let u_iv_i for $i = 1, 2, \ldots, k$ be k edges of H_2. The graph G is obtained from the union of H_1 and H_2 by first cutting each u_iv_i into two edges u_ix_i and y_iv_i and then identifying each of x_i and y_i with a vertex of H_1.

Operation B. Let u_iv_i for $i = 1, 2, \ldots, k$ be k edges of H_2. The graph G is obtained from the union of H_1 and H_2 by inserting a degree-two vertex x_i into each u_iv_i and then identifying each x_i with a vertex in H_1.

Operation C. Let u be a vertex in H_2 with $d(u) = k \geq 2$, and let $u_1, u_2, \ldots, u_t$ be t neighbors of u. The graph G is obtained from the union of H_1 and H_2 by splitting u into $t+1$ vertices $u_1', u_2', \ldots, u_t', u'$ such that u_i is the only neighbor of u_i' and that $d(u') = d(u) - t$, and then identifying each of those $t+1$ vertices with a vertex in H_1.

LEMMA 4.3. *Let H_1 be a Z_3-connected graph, and let H_2 be a cubic bipartite graph with at least four vertices. (H_2 is simple if it has at least six vertices or contains precisely two 2-circuits otherwise.) Then the graph G obtained by adding H_2 onto H_1 via Operation* A, B*, and/or* C *admits a nowhere-zero* 3*-flow.*

Remark. Obviously, the new graph G obtained via Operation C is simple as long as H_1 and H_2 are simple. Let H_3 be the graph obtained from H_2 by cutting all u_iv_i. Then the new graph G obtained via Operation A is simple if, first, H_3 and H_1 are simple; second, the edges in H_3 joining the same vertex of H_1 form a matching in H_3. If G is obtained via Operation B, then G is simple if both H_1 and the graph obtained from H_2 by inserting a new degree-two vertex into each edge u_iv_i are simple and the edges u_iv_i form a matching in H_2.

Proof. Note that H_1 remains intact in the new graph G, so it is still Z_3-connected (as a subgraph of G). By Lemma 2.5(a), G admits a nowhere-zero 3-flow if and only if G/H_1 admits a nowhere-zero 3-flow.

Since H_2 is a cubic bipartite graph, by Lemma 2.2, it admits a nowhere-zero 3-flow. Furthermore, $G/H_1 = H_2$ if only Operation C is applied, and if Operation A or B is applied, then G/H_1 can be obtained from H_2 by subdividing some edges once and then identifying the new degree-two vertices as one vertex. Therefore, G/H_1 also admits a nowhere-zero 3-flow. ∎

LEMMA 4.4. *Let $\pi = (d_1, d_2, 4^{n-4}, 3^2)$ be a sequence with $n - 1 \geq d_1 \geq d_2 \geq 4$ and $n \geq 5$. Then π is graphical, provided that $d_1 + d_2$ is even.*

Proof. Assume the contrary: π is a counterexample with minimum n. Then $n \geq 6$, for otherwise $n = 5$, so $\pi = (4^3, 3^2)$, and thus the graph obtained from K_5 by deleting one edge is a realization of π, a contradiction.

If $d_2 \geq 5$, then the residual sequence $\bar{\pi} = (d_1 - 1, d_2 - 1, 4^{n-5}, 3^2)$. From the assumption on π, we see that $\bar{\pi}$ is graphical and hence so is π, by Lemma 2.1(b); this contradiction yields $d_2 = 4$. Therefore $\pi = (d_1, 4^{n-3}, 3^2)$.

Since $d_1 + d_2 = d_1 + 4$ is even, so is d_1. It follows that the graph G^2 (resp., G^5) in the proof of Lemma 3.3 is a realization of π if $d_1 = 4$ (resp., $d_1 \geq 6$). ∎

LEMMA 4.5. *Let $\pi = (d_1, d_2, 5, 4^{n-3-k}, 3^k)$ be a graphical sequence with $n - 2 \geq d_1 \geq d_2 \geq 5$, $n - 3 \geq k \geq 5$, and $n \geq 9$. Then π has a realization G such that*

(a) *G admits a nowhere-zero 3-flow;*

(b) *G has a Z_3-connected subgraph H that contains all vertices of G with degree at least four.*

Proof. Let us distinguish between two cases according to the parity of k.

Case 1. k is even. Since $k \geq 5$ and since the degree sum of π is even, in this case we have

(1) $k \geq 6$, and $d_1 + d_2$ is odd.

We propose to construct a realization G of π with properties (a) and (b) using Lemma 4.3, such that the Z_3-connected graph H_1 (recall Lemma 4.3) has degree sequence $\pi^* = (d_1^*, d_2^*, 4^{n-2-k}, 3^2)$, where d_1^* and d_2^* are to be determined, and the cubic bipartite graph H_2 has $k - 2$ vertices.

We are to determine d_1^* and d_2^* by letting $A_i = \max\{4, d_i - (k - 2)\}$ and $B_i = \min\{n - k + 1, d_i - 1\}$ for $i = 1, 2$. Then

(2) $A_i \leq B_i$, and equality holds if and only if $A_i = B_i = 4$. To verify this, note that $4 \leq n-k+1$ (because $k \leq n-3$), $d_i-(k-2) < d_i-1$ (because $6 \leq k$), $4 \leq d_i-1$ (because $5 \leq d_i$), and $d_i - (k - 2) < n - k + 1$ (because $d_i \leq n - 2$). Combining these inequalities yields $A_i \leq B_i$ and $d_i - (k - 2) < B_i$. It follows that $A_i = B_i$ if and only if both of them are four. So (2) is true.

Clearly (2) guarantees the existence of d_1^* and d_2^* such that

- $A_i \le d_i^* \le B_i$ for $i = 1, 2$;
- $d_1^* + d_2^*$ is even.

By Lemma 4.4, the sequence $\pi^* = (d_1^*, d_2^*, 4^{n-2-k}, 3^2)$ is graphical, and hence by Theorem 3.1 it admits a Z_3-connected realization H_1. By the definitions of A_i, B_i, and d_i^*, we have $d_i - (k-2) \le d_i^* \le d_i - 1$ for $i = 1, 2$, so

(3) $1 \le d_i - d_i^* \le k - 2$.

It follows that $2 \le (d_1 + d_2) - (d_1^* + d_2^*) \le 2(k-2)$. Since $d_1 + d_2$ is odd (by (1)) while $d_1^* + d_2^*$ is even, the above inequalities can be strengthened as $3 \le (d_1 + d_2) - (d_1^* + d_2^*) \le 2(k-2) - 1$, or equivalently

(4) $2 \le \frac{(d_1+d_2)-(d_1^*+d_2^*)+1}{2} \le k - 2$.

Using Lemma 4.3, a realization of π can be constructed as follows. Let H_2 be a cubic bipartite graph with $k - 2$ (≥ 4) vertices, where H_2 is simple if $k \ge 8$ and contains precisely two disjoint 2-circuits, C_1 and C_2, if $k = 6$. Then there exist two disjoint perfect matchings M_1, M_2 in H_2 such that $M_i \cap C_i \ne \emptyset$ for $i = 1, 2$ if $k = 6$. Without loss of generality, we assume that $d_1 - d_1^*$ is even. So $d_2 - d_2^*$ is odd. By (3), (4) and the selection of M_i, we can find a subset F_i of M_i such that $|F_1| = \frac{d_1 - d_1^*}{2}$, $|F_2| = \frac{d_2 - d_2^* + 1}{2}$, and $(F_1 \cup F_2) \cap C_i \ne \emptyset$ for $i = 1, 2$ if $k = 6$. Let v_i be the vertex with degree d_i^* in H_1 for $i = 1, 2$, and let ab be a special edge in F_2. For $i = 1, 2$, let us subdivide each edge in F_i once by a degree-two vertex (let c denote this vertex on the special edge ab), then identifying all these degree-two vertices with v_i. At this stage, the resulting graph has the degree sequence $(d_1, d_2 + 1, 4^{n-2-k}, 3^k)$. Finally, switch the edge ca away from vertex v_2 to a degree-four vertex in $H_1 \setminus \{v_1, v_2\}$. Clearly, the resulting graph G is simple and is a desired realization of the sequence π.

Case 2. k is odd. In this case, we have

(5) $d_1 + d_2$ is even. According to the value of k, we consider two possibilities.

Subcase 2.1. $k \le n - 4$. The proof of this subcase goes along the same line as that of Case 1: We propose to construct a realization G of π with properties (a) and (b), using Lemma 4.3, such that the Z_3-connected graph H_1 (recall Lemma 4.3) has degree sequence $\pi^* = (d_1^*, d_2^*, 4^{n-3-k}, 3^2)$, where d_1^* and d_2^* are to be determined, and the cubic bipartite graph H_2 has $k - 1$ vertices.

We are to determine d_1^* and d_2^* by letting $A_i = \max\{4, d_i - (k-1)\}$ and $B_i = \min\{n - k,\ d_i - 1\}$ for $i = 1, 2$. It is a routine matter to check that

(6) $A_i \le B_i$, and equality holds if and only if $A_i = B_i = 4$.

Thus (6) guarantees the existence of d_1^* and d_2^* such that

- $A_i \le d_i^* \le B_i$ for $i = 1, 2$;
- $d_1^* + d_2^*$ is even.

By Lemma 4.4, the sequence $\pi^* = (d_1^*, d_2^*, 4^{n-3-k}, 3^2)$ is graphical, and hence by Theorem 3.1 it admits a Z_3-connected realization H_1. By the definitions of A_i, B_i, and d_i^*, we have $d_i - (k-1) \le d_i^* \le d_i - 1$ for $i = 1, 2$, so

(7) $1 \le d_i - d_i^* \le k - 1$.

Let H_2 be a cubic bipartite graph with $k - 1$ (≥ 4) vertices, where H_2 is simple if $k \ge 7$, or contains precisely two disjoint 2-circuits, C_1 and C_2, if $k = 5$. Renaming the subscripts if necessary, we assume that $d_1 - d_1^* \ge d_2 - d_2^*$. Set $t_1 = (d_1 - d_1^*)/2$ and $t_2 = (d_2 - d_2^*)/2$ if $d_1 - d_1^*$ is even, or, set $t_1 = (d_1 - d_1^* + 1)/2$ and $t_2 = (d_2 - d_2^* - 1)/2$ otherwise. Since $d_1 + d_2$ and $d_1^* + d_2^*$ have the same parity, t_1 and t_2 are both integers. Moreover, by (7) we have

(8) $t_2 \le t_1 \le (k-1)/2$, and $t_2 < t_1$ if $d_1 - d_1^*$ is odd.

Let M_1, M_2, M_3 be three disjoint perfect matchings in H_2. In view of (8), we can find a subset F_i of M_i such that $|F_i| = t_i$ for $i = 1, 2$, $|F_3| = 1$, and $(F_1 \cup F_2 \cup F_3) \cap C_i \neq \emptyset$ for $i = 1, 2$ if $k = 5$. Let v_i be the vertex with degree d_i^* in H_1 for $i = 1, 2$, let ab be a special edge in F_1 such that a is covered by no edge in F_2 if $d_1 - d_1^*$ is odd (such edge is available as $t_2 < t_1$), and let cd be the edge in F_3. For $i = 1, 2$, let us subdivide each edge in F_i once by a degree-two vertex (let x denote this vertex on the special edge ab) and identify all these degree-two vertices with v_i. At this stage, the resulting graph has the degree sequence $(d_1, d_2, 4^{n-3-k}, 3^{k+1})$ if $d_1 - d_1^*$ is even, or $(d_1 + 1, d_2 - 1, 4^{n-3-k}, 3^{k+1})$ if $d_1 - d_1^*$ is odd. Then switch edge xa away from vertex v_1 to v_2 if $d_1 - d_1^*$ is odd, and finally cut cd ($\in F_3$) into two edges cx and yd and identify x (resp., y) with a degree-four (resp., degree-three) vertex in $H_1 \setminus \{v_1, v_2\}$. Clearly, the resulting graph G is simple and is a desired realization of the sequence π.

Subcase 2.2. $k = n - 3$. In this subcase, by the hypothesis of the theorem we have

(9) $\pi = (d_1, d_2, 5, 3^k)$, where $n - 2 \geq d_1 \geq d_2 \geq 5$ and $n \geq 9$.

According to the hypothesis of Case 2, k is odd. From $k = n - 3$ and $n \geq 9$ we deduce that

(10) $k \geq 7$.

By Lemma 4.4, the sequence $(4^3, 3^2)$ is graphical and hence, by Theorem 3.1, it admits a Z_3-connected realization H_1; let x_1, x_2, x_3 be the three vertices of degree four in H_1. Since $k \geq 7$ by (10), we can find a simple cubic bipartite graph H_2 with $k - 1$ vertices. Let u be a vertex of H_2, let v_1, v_2, v_3 be the neighbors of u in H_2, and let M_1, M_2, M_3 be three disjoint perfect matchings of H_2. Renaming the subscripts if necessary, we assume $uv_i \in M_i$ for $i = 1, 2, 3$. Let G^* be the graph obtained from the union of H_1 and H_2 by first splitting u into three vertices $\{u_1, u_2, u_3\}$ (so the three edges incident with u in H_2 become u_1v_1, u_2v_2, u_3v_3) and then identifying u_i with x_i in H_1 for $i = 1, 2, 3$. At this stage, the resulting graph has the degree sequence $(5^3, 3^k)$.

Set $t_1 = (d_1 - 5)/2$ and $t_2 = (d_2 - 5)/2$ if d_1 is odd, and set $t_1 = (d_1 - 4)/2$ and $t_2 = (d_2 - 6)/2$ otherwise. Since $d_1 + d_2$ is even by (5), t_1 and t_2 are both integers. Moreover, since $k = n - 3$ and $n - 2 \geq d_1 \geq d_2 \geq 5$, we have

(11) $t_2 \leq t_1 \leq (k - 3)/2$, and $t_2 < t_1$ if d_1 is even.

Since $M_i - \{uv_i\}$ contains $(k - 3)/2$ edges, we can find a subset F_i of $M_i - \{uv_i\}$ such that $|F_i| = t_i$ for $i = 1, 2$. Let ab be a special edge in F_1 such that a is covered by no edge in F_2 if d_1 is even (such an edge is available as $t_2 < t_1$). We construct a graph from G^* as follows: For $i = 1, 2$, subdivide each edge in F_i once by a degree-two vertex (let c denote this vertex on the special edge ab), then identify all these degree-two vertices with x_i, and finally switch edge ca away from vertex x_1 to x_2 if d_1 is even. Clearly, the resulting graph G is simple and is a desired realization of the sequence π. □

Now we are ready to establish the main results of this section.

Proof of Theorem 4.1. Assume the contrary: π is a counterexample with minimum n. By Theorem 3.1, we have the following:

(1) $d_n = 3$.

(2) The residual sequence $\bar{\pi}$ does not satisfy the hypothesis of the theorem.

Otherwise, from the assumption on π we deduce that $\bar{\pi}$ has a realization $\bar{G}$ such that

- $\bar{G}$ admits a nowhere-zero 3-flow;
- $\bar{G}$ has a Z_3-connected subgraph $\bar{H}$ that contains all vertices of $\bar{G}$ with degree at least four.

Let G be the realization of π obtained from $\bar{G}$ by adding a new vertex v and three edges between v and corresponding vertices in $\bar{G}$ (recall (1)). Let H be the subgraph induced by $V(\bar{H}) \cup \{v\}$ in G. Since $\bar{H}$ contains all vertices of $\bar{G}$ with degree at least four, it also contains all vertices of G with degree at least four as $d_3 \geq 5$ and $d_n = 3$. So the degree of v in H is also three. Hence H is Z_3-connected by Lemma 2.6. Note that the existence of a nowhere-zero 3-flow is preserved under edge contractions, so $\bar{G}/\bar{H}$ and hence G/H (as $\bar{G}/\bar{H} = G/H$) admits a nowhere-zero 3-flow. It follows from Lemma 2.5(a) that so does G. This contradiction implies (2).

(3) $d_1 \leq n-2$. Otherwise, $d_1 = n-1$. Since $d_3 \geq 5$, π has a Z_3-connected realization by Theorem 3.2, a contradiction. So we get (3).

(4) $d_3 = 5$. Otherwise, $d_3 \geq 6$; combining this with (1), we see that $\bar{\pi}$ satisfies the hypothesis of the theorem, contradicting (2). So (4) holds.

Throughout the proof, let m_k denote the multiplicity of k in π. Then

(5) $m_3 \geq 5$. Otherwise, by Theorem 3.1, we have $m_3 = 4$. Thus precisely three entries of $\bar{\pi}$ are three by (1) and (4). It follows from Theorem 3.1 that $\bar{\pi}$ has a Z_3-connected realization. Hence so does π by Lemma 2.6. This contradiction yields (5).

(6) $n \geq 9$. Otherwise, $n \leq 8$. From (4) and (5) we deduce that $n = 8$ and $m_3 = 5$. In view of (3), $d_1 \leq 6$. So $\pi = (5^3, 3^5)$ or $(6^2, 5, 3^5)$.

For $\pi = (5^3, 3^5)$, let G be the graph obtained from the disjoint union of a W_4, with hub v_0 and rim $v_1v_2v_3v_4v_1$, and a path $v_5v_6v_7$ by adding edges $v_5v_1, v_5v_3, v_7v_1, v_7v_3, v_6v_0$. Then G/W_4 is triangularly connected and contains two 2-circuits. Since 2-circuits are Z_3-connected, so is G/W_4 by Lemma 2.9(a). It follows from Lemmas 2.9(c) and 2.5(b) that G is Z_3-connected.

For $\pi = (6^2, 5, 3^5)$, we have $\bar{\pi} = (5^2, 4, 3^4)$. Let $\bar{G}$ be the graph obtained from the disjoint union of a W_4 and a W_3 by identifying one rim edge of W_4 with a rim edge of W_3. Then $\bar{G}$ is a realization of $\bar{\pi}$. Using the same proof employed in the preceding paragraph, we can justify that $\bar{G}$ is Z_3-connected. Now let G be the graph obtained from $\bar{G}$ by adding a new vertex v and three edges between v and vertices of degree at least four in $\bar{G}$. Clearly, G is a realization of π and is Z_3-connected by Lemma 2.6. This contradiction proves (6).

From (3), (4), (5), (6), and Lemma 4.5, we deduce the following:

(7) $d_4 = 5$.

(8) $d_2 = 5$. Otherwise, $d_2 \geq 6$. In view of (7), $\bar{\pi}$ satisfies the conditions of the theorem, contradicting (2).

(9) $d_5 \leq 4$. Suppose to the contrary that $d_5 \geq 5$. By (7), we have $d_5 = 5$. It follows that $d_1 = 5$ and $d_6 \leq 4$, for otherwise $\bar{\pi}$ satisfies the conditions of the theorem, contradicting (2). Thus $\pi = (5^5, 4^{m_4}, 3^{m_3})$. Consider the sequence $\pi^* = (4^{m_4+4}, 3^2)$. By Lemma 4.4, π^* is graphical. Thus it has a Z_3-connected realization H_1 by Theorem 3.1. Note that m_3 is odd since $\pi = (5^5, 4^{m_4}, 3^{m_3})$ (by Lemma 2.1). Let H_2 be a bipartite cubic graph on $m_3 - 1$ (≥ 4) vertices, where H_2 is simple if $m_3 \geq 7$, or contains precisely two 2-circuits, C_1 and C_2, if $m_3 = 5$. We take three edges e_1, e_2, e_3 in H_2 such that e_1 and e_2 are independent and that $e_i \in C_i$ for $i = 1, 2$ if $m_3 = 5$. Let us construct a graph from the disjoint union of H_1 and H_2 as follows: Cut each of e_1 and e_2 into two edges (let u_1, u_2, u_3, u_4 denote the new vertices), then subdivide e_3 once by a degree-two vertex v, and finally identify u_1, u_2, u_3, u_4 with four degree-four vertices in H_1, respectively, and v with a degree-three vertex in H_1. Clearly, G is a realization of π. It follows from Lemma 4.3 that G admits a nowhere-zero 3-flow. This contradiction implies (9).

From the above observations, we conclude

(10) $\pi = (d_1, 5^3, 4^{n-m_3-4}, 3^{m_3})$. So $m_3 \le n-4$.

Let us distinguish between two cases according to the parity of d_1.

Case 1. d_1 is even. In view of (10), we have

(11) m_3 is odd.

We propose to construct a realization G of π with properties (a) and (b), using Lemma 4.3, such that the Z_3-connected graph H_1 (recall Theorem 3.1) has degree sequence $\pi^* = (d_1^*, 4^{n-m_3-2}, 3^2)$, where d_1^* is to be determined, and the cubic bipartite graph H_2 has $m_3 - 1$ vertices.

In order to determine d_1^*, set $A = \max\{4, d_1-(m_3-1)\}$ and $B = \min\{n-m_3, d_1\}$. By virtue of (3), (5), (10), and (11), it is a routine matter to check that

(12) $A \le B$ and equality holds if and only if $A = B = 4$.

Thus (12) guarantees the existence of d_1^* such that $A \le d_1^* \le B$ and that d_1^* is even. By Lemma 4.4, the sequence $\pi^* = (d_1^*, 4^{n-m_3-2}, 3^2)$ is graphical, and hence by Theorem 3.1 it admits a Z_3-connected realization H_1. By the definitions of A, B, and d_1^*, we have $d_1 - (m_3-1) \le d_1^* \le d_1$, so

(13) $0 \le d_1 - d_1^* \le m_3 - 1$, and hence $0 \le \frac{d_1-d_1^*}{2} \le \frac{m_3-1}{2}$ (recall Case 1).

Let H_2 be a cubic bipartite graph with m_3-1 (≥ 4) vertices, where H_2 is simple if $m_3 \ge 7$ and contains precisely two disjoint 2-circuits, C_1 and C_2, if $m_3 = 5$ (see (11)). By (13), there exist two disjoint matchings M_1, M_2 in H_2 such that $|M_1| = \frac{d_1-d_1^*}{2}$, $|M_2| = 2$, and $(M_1 \cup M_2) \cap C_i \ne \emptyset$ for $i = 1, 2$ if $m_3 = 5$. Let us first subdivide each edge in M_1 once by a new vertex and identify all these vertices with the vertex of degree d_1^* in H_1, and then cut each edge in M_2 into two edges (let v_1, v_2, v_3, v_4 be the new vertices) and identify v_1, v_2, v_3 with three degree-four vertices in H_1, respectively, and v_4 with a degree-three vertex in H_2. Clearly, the resulting graph G is simple and is a desired realization of the sequence π. By Lemma 4.3, G admits a nowhere-zero 3-flow.

Case 2. d_1 is odd. From (10) we see that

(14) m_3 is even. So $m_3 \ge 6$ by (5).

We propose to construct a realization G of π with properties (a) and (b), using Lemma 4.3, such that the Z_3-connected graph H_1 (recall Theorem 3.1) has degree sequence $\pi^* = (d_1^*, 4^{n-m_3-1}, 3^2)$, where d_1^* is to be determined, and the cubic bipartite graph H_2 has $m_3 - 2$ vertices.

To this end, set $A = \max\{4, d_1 - (m_3 - 2)\}$ and $B = \min\{n - m_3 + 1, d_1\}$. By virtue of (3), (5), (10), and (14), we get

(15) $A \le B$ and equality holds if and only if $A = B = 4$.

Thus (15) guarantees the existence of d_1^* such that $A \le d_1^* \le B$ and that d_1^* is even. By Lemma 4.4, the sequence $\pi^* = (d_1^*, 4^{n-m_3-1}, 3^2)$ is graphical, and hence by Theorem 3.1 it admits a Z_3-connected realization H_1. By the definitions of A, B, and d_1^*, we have $d_1 - (m_3 - 2) \le d_1^* \le d_1$, so

(16) $0 \le d_1 - d_1^* \le m_3 - 2$, and hence $1 \le \frac{d_1-d_1^*+1}{2} \le \frac{m_3-2}{2}$ (as d_1 and d_1^* have different parities).

Let H_2 be a cubic bipartite graph with $m_3 - 2$ (≥ 4) vertices, where H_2 is simple if $m_3 \ge 8$, or contains precisely two disjoint 2-circuits, C_1 and C_2, if $m_3 = 6$ (see (14)). By (16), there exist a pair of edge-disjoint matchings M_1, M_2 in H_2 such that $|M_1| = \frac{d_1-d_1^*+1}{2}$, $|M_2| = 1$, and $(M_1 \cup M_2) \cap C_i \ne \emptyset$ for $i = 1, 2$ if $m_3 = 6$. Let ab be a special edge in M_1. We construct a graph from the disjoint union of H_1 and H_2 as follows: First subdivide each edge in M_1 once by a new vertex (let x be the new vertex on the special ab) and identify all these new vertices with the vertex v_1 of degree d_1^* in H_1, then cut the edge in M_2 into two edges (let y, z be the new vertices) and identify y, z with two degree-four vertices in H_1, respectively. At this stage, the resulting graph has the degree sequence $(d_1+1, 5^2, 4^{n-m_3-3}, 3^{m_3})$. Finally, switch the edge ax away from v_1 to some degree-four vertex. Clearly, the resulting graph G is simple and is a desired realization of the sequence π (see (10)). By Lemma 4.3, G admits a nowhere-zero 3-flow. This contradiction completes the proof of the theorem. □

Proof of Theorem 4.2. The proof goes along the same line as that of Lemma 4.5, so we give only a sketch here. Let us consider four cases according to the values of k and d_4: In each case we propose to construct a realization G of π with properties (a) and (b) using Lemma 4.3; the degree sequence π^* of the Z_3-connected graph H_1 (recall Lemma 4.3) and the number of vertices in the cubic bipartite graph H_2 are given below.

Case 1. k is even and at least six. The number of vertices in H_2 is $k-2$, and the degree sequence π^* of H_1 is $(d_1^*, d_2^*, 4^{n-2-k}, 3^2)$. In order to determine d_1^* and d_2^*, we

- set $A_i = \max\{4, d_i - (k-2)\}$ for $i = 1, 2$,
- set $B_1 = \min\{n-k+1,\ d_1 - p\}$, where $p = 3$ if $d_2 = 4$ and 2 if $d_2 \geq 5$, and
- set $B_2 = \min\{n-k+1,\ d_2 - q\}$, where $q = 0$ if $d_2 = 4$ and 1 if $d_2 \geq 5$.

Case 2. $k = 4$. The number of vertices in H_2 is k, and the degree sequence π^* of H_1 is $(d_1^*, d_2^*, 4^{n-4-k}, 3^2)$. In order to determine d_1^* and d_2^*, we

- set $A_i = \max\{4, d_i - k\}$ for $i = 1, 2$,
- set $B_1 = \min\{n-k-1,\ d_1 - p\}$, where $p = 3$ if $d_2 = 4$ and 2 if $d_2 \geq 5$, and
- set $B_2 = \min\{n-k-1,\ d_2 - q\}$, where $q = 0$ if $d_2 = 4$ and 1 if $d_2 \geq 5$.

Case 3. k is odd and $d_4 = 4$. The number of vertices in H_2 is $k-1$, and the degree sequence π^* of H_1 is $(d_1^*, d_2^*, 4^{n-3-k}, 3^2)$. From $d_4 = 4$ it can be seen that $n-3-k \geq 1$. In order to determine d_1^* and d_2^*, we

- set $A_i = \max\{4, d_i - (k-1)\}$ for $i = 1, 2$,
- set $B_1 = \min\{n-k,\ d_1 - p\}$, where $p = 3$ if $d_2 = 4$ and 2 if $d_2 \geq 5$, and
- set $B_2 = \min\{n-k,\ d_2 - q\}$, where $q = 0$ if $d_2 = 4$ and 1 if $d_2 \geq 5$.

In each of the above three cases, it is a routine matter to check that $A_i \leq B_i$ and equality holds if and only if $A_i = B_i = 4$. Thus there exist d_1^* and d_2^* such that

- $A_i \leq d_i^* \leq B_i$ for $i = 1, 2$;
- $d_1^* + d_2^*$ is even.

By Lemma 4.4, the sequence π^* is graphical, and hence by Theorem 3.1 it admits a Z_3-connected realization H_1. Clearly, we can choose H_2 so that it is simple if it has at least six vertices and contains precisely two disjoint 2-circuits otherwise. Note that H_2 contains three disjoint perfect matchings. By subdividing or cutting a certain number (at least two) of edges in these matchings, we can get a realization G of π, as desired.

Case 4. k is odd and $d_4 = 3$. In this case $n-k-2 = 1$, so $k = n-3$. Hence $\pi = (d_1, d_2, 4, 3^k)$, $k \geq 7$, and n is even (recall the hypothesis of the theorem). By Lemma 2.1(a), $d_1 + d_2$ is odd. From $d_1 + d_2 \geq 11$ we further deduce that $n-3 \geq d_1 \geq 7$ if d_1 is odd.

Let H_1 be a Z_3-connected realization of $(4^3, 3^2)$ in which vertices x_1, x_2, x_3 are of degree four, and let H_2 be a cubic bipartite graph with $k-1$ vertices. Let u be a vertex of H_2, let v_1, v_2, v_3 be the neighbors of u in H_2, and let M_1, M_2, M_3 be three disjoint perfect matchings of H_2. Renaming the subscripts if necessary, we assume $uv_i \in M_i$ for $i = 1, 2, 3$. For odd d_1, let G^* be the graph obtained from the union of H_1 and H_2 by identifying x_1 and u. For even d_1, let G^* be the graph obtained from the union of H_1 and H_2 by first splitting u into three vertices $\{u_1, u_2, u_3\}$ (so the three edges incident with u in H_2 become u_1v_1, u_2v_2, u_3v_3) and then identifying u_1, u_2 with x_1 and u_3 with x_2.

Set $t_1 = (d_1 - 7)/2$ and $t_2 = (d_2 - 4)/2$ if d_1 is odd, and set $t_1 = (d_1 - 6)/2$ and $t_2 = (d_2 - 5)/2$ otherwise. Since $d_1 + d_2$ is odd, t_1 and t_2 are both integers. Moreover, since $k = n - 3$, $n - 2 \geq d_1 \geq d_2 \geq 4$, and $d_1 \leq n - 3$ if d_1 is odd, we have

- $t_1 \leq (k-1)/2 - 3$ if d_1 is odd and $t_1 \leq (k-1)/2 - 2$ otherwise, and
- $t_2 \leq (k-1)/2 - 1$ if d_1 is odd and $t_2 \leq (k-1)/2 - 1$ otherwise.

For odd d_1, let F_1 be a subset of M_1 such that F_1 covers none of v_1, v_2, v_3 and $|F_1| = t_1$, and let F_2 be a subset of M_2 such that $uv_2 \notin F_2$ and $|F_2| = t_2$. For even d_1, let F_1 be a subset of M_1 such that F_1 covers neither v_1 nor v_2 and $|F_1| = t_1$, and let F_2 be a subset of M_3 such that $uv_3 \notin F_2$ and $|F_2| = t_2$. Finally, we construct a graph from G^* as follows: For $i = 1, 2$, subdivide each edge in F_i once by a degree-two vertex, and then identify all these degree-two vertices with x_i. Clearly, the resulting graph G is simple and is a realization of π that admits a nowhere-zero 3-flow, a contradiction. ∎

5. Realizations with 3-flows. To establish the main theorem of this paper, we shall break the proof into two parts and turn to proving the following two theorems.

THEOREM 5.1. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 3$. Then π has a realization G that admits a nowhere-zero 3-flow if and only if $\pi \neq (k, 3^k)$, $(k^2, 3^{k-1})$, where k is odd.*

THEOREM 5.2. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence with $d_1 \geq d_2 \geq \cdots \geq d_n = 2$. Then π has a realization G that admits a nowhere-zero 3-flow if and only if $\pi \neq (3^4, 2)$.*

The following lemma will be used repeatedly in our proof.

LEMMA 5.3. *Let $k \geq 4$ be an even integer, and let $\pi = (a_1, a_2, a_3, 3^k)$ be a sequence with $0 \leq a_i \leq k$. Then π has a realization that admits a nowhere-zero 3-flow if one of the following holds:*

(a) *a_i is even for $i = 1, 2, 3$, and $a_1 + a_2 + a_3 \geq 4$ if $k = 4$;*

(b) *one of a_1, a_2, a_3 is even, and the remaining two are odd and at least three.*

Proof. (a) Let H be a cubic bipartite graph with k vertices, where H is simple if $k \geq 6$, or contains precisely two disjoint 2-circuits, C_1 and C_2, if $k = 4$. Then the edge set of H can be decomposed into three perfect matchings M_1, M_2, M_3. Since $a_i \leq k$ and since $a_1 + a_2 + a_3 \geq 4$ if $k = 4$, we can find a subset F_i of M_i such that $|F_i| = \frac{a_i}{2}$ for $i = 1, 2, 3$ and that $(F_1 \cup F_2 \cup F_3) \cap C_i \neq \emptyset$ for $i = 1, 2$ if $k = 4$. For $i = 1, 2, 3$, let us subdivide each edge in F_i once by a degree-two vertex and then identify all these degree-two vertices as a single vertex v_i. Then the resulting graph G is a realization of the sequence π. By Lemma 2.2, H admits a nowhere-zero 3-flow, and so does G.

(b) Renaming the subscripts if necessary, we may assume that a_3 is even. By Lemma 2.1, k is even and so $a_i \leq k - 1$ for $i = 1, 2$. Let H be a simple cubic bipartite graph with $k + 2$ vertices, and let M_1, M_2, M_3 be three disjoint perfect matchings of H. Take an arbitrary edge u_1v_1 in M_1, and take a subset F_i of M_i such that $|F_i| = \frac{a_i - 3}{2}$ for $i = 1, 2$, $|F_3| = \frac{a_3}{2}$, and F_1 (resp., F_2) covers no vertices of $N(u_1)$, the neighborhood of u_1 (resp., $N(v_1)$). Let us now subdivide each edge of F_1 (resp., F_2) once by a degree-two vertex and then identify all these degree-two vertices with u_1 (resp., v_1), and subdivide each edge of F_3 once by a degree-two vertex and then identify all these degree-two vertices. Then the resulting graph G is a realization of the sequence π. By Lemma 2.2, H admits a nowhere-zero 3-flow, and so does G. ∎

Proof of Theorem 5.1. Since the "only if" part is already established by Lemma 2.10, let us proceed to the "if" part.

Assume the contrary: π is a counterexample with minimum n. By Theorems 3.1, we have

(1) $m_3 \geq 4$.

From Theorem 4.1 it can be seen that

(2) $d_3 \leq 4$.

Let us further make some simple observations.

(3) $4 \leq d_1 \leq n-2$. The upper bound follows instantly from Theorem 3.2. To justify lower bound, we assume to the contrary that $d_1 = 3$. So $\pi = (3^n)$. Using the upper bound, we have $n \geq 5$. From Lemma 2.1(a) we deduce that n is even and so $n \geq 6$. Thus π can be realized by a bipartite cubic graph, which, by Lemma 2.2, admits a nowhere-zero 3-flow, a contradiction. So (3) holds.

(4) $d_2 \geq 4$. Otherwise, $d_2 = 3$. Thus $\pi = (d_1, 3^{n-1})$. If $n-1$ is even, then so is d_1. Set $a_1 = d_1$, $a_2 = a_3 = 0$, and $k = n-1$. By (3) and Lemma 5.3(a), π has a realization that admits a nowhere-zero 3-flow, a contradiction.

So we assume that $n-1$ is odd. In this case d_1 is also odd. Write $\pi = (d_1, 3, 3^{n-2})$. Set $a_1 = 0$, $a_2 = d_1$, $a_3 = 3$, and $k = n-2$. Then, by (3) and Lemma 5.3(b), π has a realization that admits a nowhere-zero 3-flow, a contradiction. This proves (4).

(5) $d_3 = 4$. By (2), $d_3 \leq 4$. We prove by contradiction and assume $d_3 = 3$. Thus $\pi = (d_1, d_2, 3^{n-2})$. Observe that $d_1 + d_2$ is odd, for otherwise (1), (3), and Lemma 5.3 (with $a_i = d_i$ for $i = 1, 2$ and $a_3 = 0$) would guarantee the existence of a realization of π that admits a nowhere-zero 3-flow, a contradiction. It follows that n and exactly one of d_1 and d_2 are odd.

If $d_1 \leq n-3$, then, by Lemma 5.3(b) (with $a_i = d_i$ for $i = 1, 2$, $a_3 = 3$, and $n-3 \geq 4$ because n is odd), π has a realization that admits a nowhere-zero 3-flow; this contradiction implies that $d_1 = n-2$. Hence d_1 is odd and d_2 is even. Now let us take odd wheel W_{n-2} with hub v_0 and rim $v_1 v_2 \ldots v_{n-2} v_1$ and take $M = \{v_{2i} v_{2i+1} : 0 \leq i \leq d_2/2 - 1\}$. Clearly, M is a matching of size $d_2/2$. Let us subdivide each edge in M once and identify all the new vertices as a single vertex. Then the resulting graph is a realization of π and admits a nowhere-zero 3-flow (to find it, direct each of edges $v_0 u$, $v_0 v_2$, $v_0 v_3$ from v_0 to the other end, where u is the vertex subdividing $v_0 v_1$; then directions of the remaining edges can be determined accordingly). This contradiction implies (5).

(6) $n \geq 9$. Otherwise, $n \leq 8$. By (5) and (1), we have $d_3 = 4$ and $m_3 \geq 4$. So $n \geq 7$.

If $n = 7$ then, by (3), (5), and Lemma 2.1(a), we have $\pi = (4^3, 3^4)$ or $(5^2, 4, 3^4)$. For $\pi = (4^3, 3^4)$, clearly $K_{3,4}$ is a realization of π that admits a nowhere-zero 3-flow. For $\pi = (5^2, 4, 3^4)$, let G be the graph obtained from the union of W_4 and W_3 by identifying a rim edge of W_4 with a rim edge of W_3. Then G is a realization of π that, by Lemma 2.9(c) and (a), admits a nowhere-zero 3-flow. So we have $n = 8$.

Since $d_3 = 4$ by (5), we have $m_3 \leq 5$; combining this with (1), we further have $m_3 = 4$ or 5. Since $d_1 \leq 6$ by (3), one of the following cases must occur:

- $\pi = (6^2, 4^2, 3^4)$,
- $\pi = (6, 4^3, 3^4)$,
- $\pi = (4^4, 3^4)$,
- $\pi = (5^2, 4^2, 3^4)$,
- $\pi = (5, 4^2, 3^5)$, or
- $\pi = (6, 5, 4, 3^5)$.

For each π, we shall exhibit a realization G that admits a nowhere-zero 3-flow, thereby reaching a contradiction.

For $\pi = (6^2, 4^2, 3^4)$, let $\bar{G}$ be the graph obtained from the union of W_4 and W_3 by identifying a rim edge of W_4 with a rim edge of W_3. Then $\bar{G}$ is the realization of the residual sequence $\bar{\pi}$ and $\bar{G}$ is Z_3-connected. So it is easy to obtain a realization G of π from $\bar{G}$ such that G admits a nowhere-zero 3-flow, a contradiction.

For $\pi = (6, 4^3, 3^4)$, let G be the graph obtained from W_6 by adding an edge between two nonadjacent vertices, then subdividing two independent edges once each, and finally identifying these new vertices as one vertex. (Since W_6 is Z_3-connected, so is the graph obtained from W_6 by adding an edge by Lemma 2.7.)

For $\pi = (4^4, 3^4)$, let G be the graph obtained from the cubic bipartite graph with four vertices by subdividing each of the four multiple edges once and then connecting these four degree-two vertices with a 4-circuit. (Note that G can be decomposed into a subdivision of a cubic bipartite graph and a 4-circuit.)

For $\pi = (5^2, 4^2, 3^4)$, let G be the graph obtained from the union of two W_4's by identifying a rim edge of one W_4 with a rim edge of the other W_4. (In fact G is Z_3-connected.)

For $\pi = (5, 4^2, 3^5)$, let G be the graph obtained from a $K_{3,3}$ (with color classes $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$) by subdividing $u_i v_i$ once with a degree-two vertex w_i for $i = 2, 3$, and then adding a triangle $u_1 w_2 w_w u_1$.

For $\pi = (6, 5, 4, 3^5)$, let G be the graph obtained from W_4 (in which $u_1 u_2$ is a rim edge) by adding a path $v_1 v_3 v_2$ and then adding edges $v_1 u_1, v_1 u_2, v_2 u_1, v_2 u_2, v_3 u_1$. (Note that G is triangularly connected and contains W_4. So it is Z_3-connected, by Lemma 2.9(a).)

It is a routine matter to check that G is a realization of π and admits a nowhere-zero 3-flow in each case. This contradiction implies (6).

From (3), (5), (6), and Theorem 4.2, we deduce that

(7) $d_1 + d_2 \leq 10$.

Since $d_1 \geq d_2 \geq 4$, we see that (d_1, d_2) is $(6, 4)$, or $(5, 5)$, or $(5, 4)$, or $(4, 4)$. So the following is the complete list of all possible configurations of π:

- $\pi = (6, 4^{m_4}, 3^{m_3})$,
- $\pi = (5^2, 4^{m_4}, 3^{m_3})$,
- $\pi = (5, 4^{m_4}, 3^{m_3})$, and
- $\pi = (4^{m_4}, 3^{m_3})$,

where m_k is the multiplicity of k in π. Let us process these cases one by one: For each π, we shall construct a realization G that admits a nowhere-zero 3-flow, thereby reaching a contradiction.

Case 1. $\pi = (4^{m_4}, 3^{m_3})$. Note that m_3 is even. Depending on the value of m_3, we consider two subcases.

Subcase 1.1. $m_3 = 4$. Our proof relies on the following statement.

(8) The sequence $(4^k, 2^4)$, with $k \geq 1$, can be realized by a simple connected graph H that admits a nowhere-zero 3-flow. To justify this, we apply induction on k. For $k = 1$, the graph H_1 obtained from two triangles by gluing them at a common vertex is as desired. Suppose that H_k is the desired realization of $(4^k, 2^4)$. Let e, f be two independent edges in H_k, and let H_{k+1} be the graph obtained from H_k by first subdividing each of e, f once with a degree-two vertex and then identifying these degree-two vertices. Clearly, H_{k+1} is a realization of $(4^{k+1}, 2^4)$ and admits a nowhere-zero 2-flow since it is Eulerian. So (8) holds.

By (6), we have $m_4 \geq 5$. In view of (8), we can find a connected realization H of the sequence $(4^{m_4-4}, 2^4)$. Let G be the graph obtained from H by adding a bipartite cubic graph F with four vertices, then subdividing each of the four multiple edges in

F with a degree-two vertex, and finally identifying these four degree-two vertices with the degree-two vertices of H, respectively. Clearly, G is a realization of π and admits a nowhere-zero 3-flow.

Subcase 1.2. $m_3 \geq 6$. For $m_4 \geq 5$, let H_1 be a bipartite cubic simple graph with m_3 vertices, and let H_2 be a Z_3-connected realization of $(4^{m_4-2}, 3^2)$ (recall Lemmas 4.4 and 3.3). Then we can get a desired realization G of π by cutting one edge of H_1 and then connecting these two half edges to the degree-three vertices of H_2, respectively. For $3 \leq m_4 \leq 4$, let G be the graph obtained from the union of H_1 and an m_4-circuit H_3 by subdividing m_4 edges of H_1 once with degree-two vertices and then identifying these new vertices with m_4 vertices of H_3, respectively. Clearly, G is a realization of π and admits a nowhere-zero 3-flow.

Case 2. $\pi = (5, 4^{m_4}, 3^{m_3})$. Note that m_3 is odd, so $m_3 \geq 5$ by (1). We distinguish two subcases according to the value of m_3.

Subcase 2.1. $m_3 = 5$. Recall that $n \geq 9$ by (6). For $n = 9$, let H_1 be the graph obtained from W_3 by adding a new vertex and joining it to two vertices of the W_3. Clearly, H_1 admits a nowhere-zero 3-flow and has degree sequence $(4^2, 3^2, 2)$. For $n \geq 10$, we have $n - m_3 - 2 \geq 3$. By Lemma 4.4, the sequence $(4^{n-m_3-2}, 3^2)$ is graphical and hence, by Lemma 3.3, admits a Z_3-connected realization F. Let H_1 be the graph obtained from F by subdividing one edge once. Clearly, H_1 admits a nowhere-zero 3-flow and has degree sequence $(4^{n-m_3-2}, 3^2, 2)$. Let H_2 be the cubic bipartite graph on four vertices in which both u_1v_1 and u_2v_2 are of multiplicity two, and let G be the graph obtained from the union of H_1 and H_2 by subdividing u_iv_i once for $i = 1, 2$, and then identifying one new vertex with the degree-two vertex of H_1 and the other new vertex with a degree-three vertex of H_1. Clearly, G is a realization of π and admits a nowhere-zero 3-flow.

Subcase 2.2. $m_3 \geq 7$. Since $d_3 = 4$ by (5), we have $m_4 \geq 2$. Let H_1 be a realization of $(4^{n-m_3-2}, 3^2, 2)$ as exhibited in the preceding paragraph, and let H_2 be a cubic bipartite simple graph with $m_3 - 1$ vertices. Using H_1 and H_2 and following the same argument as the preceding paragraph, we can obviously get a realization G of π that admits a nowhere-zero 3-flow.

Case 3. $\pi = (6, 4^{m_4}, 3^{m_3})$. Let H be an arbitrary realization of π, and let u be the vertex of degree six in H. Then the configuration of π implies the existence of two nonadjacent neighbors v, w of u in H. Let H' be the graph obtained from H by replacing path vuw with edge vw. Then the degree sequence of H' is $\pi' = (4^{m_4+1}, 3^{m_3})$. By Case 1, π' has a realization G' that admits a nowhere-zero 3-flow. Moreover, if $\pi' = (4^5, 3^4)$, by Subcase 1.1, G' can be chosen such that there is a degree-four vertex x and an edge e such that x is not incident with e and is not adjacent to the end-vertices of e. Let x be a degree-four vertex in G'. Let $X_1 = N(x) \cup \{x\}$ and $X_2 = V(G') \setminus X_1$. Then there must be an edge e not incident with x such that x is not adjacent to the end-vertices of e if $\pi' \neq (4^5, 3^4)$. Otherwise, G' is connected and X_2 is an independent set. Let $[X_1, X_2]$ denote the set of edges with one end in X_1 and the other in X_2. Then $(n-5) \times 3 \leq \sum_{u \in X_2} d(u) = |[X_1, X_2]| \leq \sum_{v \in X_1 \setminus \{x\}} (d(v) - 1) \leq 4 \times 3 = 12$. Hence, $n \leq 9$ with equality if and only if X_1 consists of all degree-four vertices and X_2 consists of all degree-three vertices. By (6), we have $n = 9$. Therefore, the degree sequence of G' is $(4^5, 3^4)$, a contradiction to the assumption that $\pi' \neq (4^5, 3^4)$. Therefore, in any case, we can find a degree-four vertex a and an edge e such that a is not incident with e and is not adjacent to the end-vertices of e. Let G be the graph obtained from G' by subdividing e once and then identifying the new vertex with a. Clearly, G is a realization of π and admits a nowhere-zero 3-flow.

Case 4. $\pi = (5^2, 4^{m_4}, 3^{m_3})$. Since π is graphical, using the same argument employed in the preceding paragraph we deduce that the sequence $(5, 4^{m_4}, 3^{m_3+1})$ is graphical and hence, by Case 2, has a realization H that admits a nowhere-zero 3-flow. Let x be a degree-three vertex. Let $X_1 = N(x) \cup \{x\}$ and $X_2 = V(H) \setminus X_1$. We first show that there exists an edge e such that x is not incident with e and is not adjacent to the end-vertices of e. Otherwise, H is connected and X_2 is independent. Let $[X_1, X_2]$ denote the set of edges with one end in X_1 and the other in X_2. Since the degree of each vertex in X_2 is at least three and the degree of each vertex in X_1 is at most five, we have $(n-4) \times 3 \le \sum_{u \in X_2} d(u) = |[X_1, X_2]| \le \sum_{v \in X_1 \setminus \{x\}} (d(v) - 1) \le 4 \times 3 = 12$. Therefore, $n \le 4$, contradicting (6), i.e., that $n \ge 9$. Let e be an edge and x be a degree vertex not incident with e such that x is not adjacent to any end-vertices of e. Let G be the graph obtained from H by subdividing an edge e once and then identifying the new vertex with a degree-three vertex not incident to e. Clearly, G is a realization of π and admits a nowhere-zero 3-flow. This completes the proof of Theorem 5.1. □

Let us make some preparation before presenting the proof of Theorem 5.2.

LEMMA 5.4. *Let k be an integer with $k = 2$ or $k \ge 4$, and let $\pi = (k, 3^k, 2)$ or $(k^2, 3^{k-1}, 2)$. If π is graphical, then it has a realization that admits a nowhere-zero 3-flow.*

Proof. Note that if $k = 2$, then $\pi = (3^2, 2^2)$. Let G be the graph obtained from W_3 by deleting one edge. Clearly, G is a realization of π and admits a nowhere-zero 3-flow. So we assume

(1) $k \ge 4$. According to the configurations of π, we consider two cases.

Case 1. $\pi = (k, 3^k, 2)$. If k is even, then, by Lemma 5.3(a), π has a realization that admits a nowhere-zero 3-flow. It remains to consider the subcase when k is odd. Thus $k \ge 5$ by (1). Let H be a bipartite cubic simple graph with $k+1$ vertices, let u be a vertex of H, and let $\{v_1, v_2, v_3\}$ be the neighbors of u. Then $H \setminus \{u, v_1, v_2\}$ contains a matching M of size $(k-3)/2$. Let G be the graph obtained from H by subdividing each edge in M once, then identifying all the degree-two vertices with u, and finally subdividing one edge uv_3 once. Clearly, G is a realization of π and admits a nowhere-zero 3-flow.

Case 2. $\pi = (k^2, 3^{k-1}, 2)$. In this case k is odd, so $k \ge 5$ by (1). Write $k = 2t+1$. Let H be the graph obtained from the disjoint union of a 4-circuit and $t-1$ triangles by gluing them at a common vertex x. Then H has $2t+2$ vertices and degree sequence $(2t, 2^{2t+1})$. Let G be the graph obtained from H by adding a new vertex y and making it adjacent to all vertices of H except precisely one degree-two vertex in a triangle. Then the degree sequence of G is $((2t+1)^2, 3^{2t}, 2)$, which is exactly π. Since G is triangularly connected and contains W_4, it is Z_3-connected by Lemma 2.9 and hence admits a nowhere-zero 3-flow. □

Proof of Theorem 5.2. The "only if" part is already established by Lemma 2.10, so we proceed to the "if" part.

Assume the contrary: π is a counterexample with minimum n. Observe that

(1) $d_2 \ge 3$. Otherwise, $d_2 = d_3 = \cdots = d_n = 2$. So d_1 is even. Thus each realization of π admits a nowhere-zero 2-flow; this contradiction leads to (1).

(2) The sequence $\sigma = (d_1, d_2, \ldots, d_{n-1})$ is not graphical. Assume to the contrary that σ is graphical. Then $\sigma \ne (3^4, 2)$, $(k, 3^k)$, $(k^2, 3^{k-1})$, where k is an odd integer, for otherwise $\pi = (3^4, 2^2)$, $(k, 3^k, 2)$, or $(k^2, 3^{k-1}, 2)$, so π has a realization that admits a nowhere-zero 3-flow by Lemma 5.3(a) or Lemma 5.4, a contradiction. By Theorem 5.1 and the assumption on π, the sequence σ has a realization H that admits a nowhere-zero 3-flow. Let G be a graph obtained from H by subdividing an edge once. Clearly,

G is a realization of π and admits a nowhere-zero 3-flow; this contradiction implies (2).

(3) Let $\bar{G}$ be an arbitrary realization of the residual sequence $\bar{\pi} = (d_1 - 1, d_2 - 1, d_3, \ldots, d_{n-1})$ and let v_i be a vertex of $\bar{G}$ with degree $d_i - 1$ for $i = 1, 2$. Then v_1v_2 is an edge of $\bar{G}$.

Otherwise, v_1 and v_2 are nonadjacent in $\bar{G}$. Thus the graph obtained from $\bar{G}$ by adding edge v_1v_2 is a realization of the sequence $\sigma = (d_1, d_2, \ldots, d_{n-1})$, contradicting (1). So (3) holds.

(4) The residual sequence $\bar{\pi}$ is $(3^4, 2)$, $(k, 3^k)$, or $(k^2, 3^{k-1})$, where k is an odd integer.

Otherwise, $\bar{\pi}$ has a realization $\bar{G}$ that admits a nowhere-zero 3-flow. Let v_i be a vertex of $\bar{G}$ with degree $d_i - 1$ for $i = 1, 2$. By (4), v_1v_2 is an edge of $\bar{G}$. Let G be the graph obtained from $\bar{G}$ by adding a new vertex w and making it adjacent to both v_1 and v_2. Since G contains the triangle wv_1v_2w and since $\bar{G}$ admits a nowhere-zero 3-flow, it is easy to see that so does G. Hence (4) is justified.

From (4) we deduce that one of the following four cases must occur:

- $\pi = (4, 3^4, 2)$,
- $\pi = (4^2, 3^2, 2^2)$,
- $\pi = (k + 1, 4, 3^{k-1}, 2)$ or
- $\pi = ((k + 1)^2, 3^{k-1}, 2)$,

where k is an odd integer. In each case we shall construct a realization of π that admits a nowhere-zero 3-flow, thereby reaching a contradiction.

For $\pi = (4, 3^4, 2)$, π has a realization G obtained by subdividing one edge once of a W_4. Since W_4 admits a nowhere-zero 3-flow, so does G.

For $\pi = (4^2, 3^2, 2^2)$, let G be the graph obtained from a W_3 by adding a new vertex, making it adjacent to two vertices of the W_3, and then subdividing an edge. Clearly G is a realization of π. To see that G admits a nowhere-zero 3-flow, let H be the graph obtained from W_3 by duplicating an edge. Then H is triangularly connected and contains a 2-circuit. By Lemma 2.9, H is Z_3-connected. So G admits a nowhere-zero 3-flow as it is a subdivision of H.

For $\pi = (k + 1, 4, 3^{k-1}, 2)$, let G be a graph obtained from W_k by adding a new vertex and making it adjacent to the hub and a rim vertex. It is easy to see that G is a realization of π and admits a nowhere-zero 3-flow.

For $\pi = ((k + 1)^2, 3^{k-1}, 2)$, let G be the graph obtained from the disjoint union of $\frac{k-1}{2}$ copies of W_3 by gluing all of them along an edge uv, and then adding a new vertex and making it adjacent to both u and v. Clearly, G is a realization of π and admits a nowhere-zero 3-flow.

This completes the proof of Theorem 5.2 and hence of Theorem 1.2. □

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