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NOWHERE-ZERO 4-FLOWS AND CAYLEY GRAPHS ON SOLVABLE GROUPS*

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Abstract. We prove that every Cayley graph on a finite solvable group admits a nowhere-zero 4-flow. In particular, every cubic Cayley graph on a solvable group is 3-edge-colorable.

Key words. integer flow, edge-coloring, Cayley graph

AMS subject classifications. 05C25, 05C15

1. Introduction. Throughout this paper graphs have neither loops nor multiple edges. We use the term *multigraph* when multiple edges are allowed. If X is a graph, $V(X)$ and $E(X)$ denote the vertex set and edge set, respectively, of X.

DEFINITION 1.1. Let X be a graph and $D(X)$ be an orientation of X. A *k-flow* on X is an integer-valued function $f : E(X) \to (-k, k)$ such that for every vertex $u \in V(X)$ the sum of the flow values on the outgoing arcs from u in $D(X)$ equals the sum of the flow values on the incoming arcs at u in $D(X)$. If $f(e) \neq 0$ for every edge $e \in E(X)$, the flow is called a *nowhere-zero* k-flow.

There are several well-known unsolved problems related to flow problems. Probably the best known unsolved problem dealing with flows is the following problem of Tutte [13].

CONJECTURE 1.2. Every 2-connected graph containing no subdivision of the Petersen graph admits a nowhere-zero 4-flow. F. Jaeger [6] proved the first of the following two results. The second of the two is a consequence of the four-color theorem.

THEOREM 1.3. *Every 4-edge-connected graph admits a nowhere-zero 4-flow.*

THEOREM 1.4. *Every 2-edge-connected planar graph admits a nowhere-zero 4-flow.*

DEFINITION 1.5. Let G be a finite group and $S \subset G$ satisfy $1 \notin S$ and $s \in S$ if and only if $s^{-1} \in S$. The *Cayley graph* $X(G; S)$ is the graph with vertex set G and $ab \in E(G)$ if and only if $b = as$ for some $s \in S$.

The first of the following two conjectures was originally posed by L. Lovász [9] as a research problem and has come to be known as Lovász's conjecture. The consideration of Lovász's conjecture quickly led a number of people to consider the second of the two. It has been attributed to various people in the literature, but it is not at all clear who initially posed it.

CONJECTURE 1.6. Every connected vertex-transitive graph has a Hamilton path.

CONJECTURE 1.7. Every connected Cayley graph with three or more vertices contains a Hamilton cycle.

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These two conjectures have attracted considerable attention over the last 24 years, and there have been many partial results. Some of the partial results are the results of very nice work; nevertheless, in many ways very little is really known about resolving the two conjectures. Since a graph with a Hamilton cycle admits a nowhere-zero 4-flow, in order for Conjecture 1.7 to be true it must be the case that appropriate Cayley graphs admit nowhere-zero 4-flows. This led Alspach and Zhang to make the following weaker conjecture at the Louisville workshop on Hamilton cycles in 1992.

CONJECTURE 1.8. Every Cayley graph with degree at least two admits a nowhere-zero 4-flow.

Another motivation for the preceding conjecture is that every graph admitting a nowhere-zero 4-flow admits a cycle double cover (see [7], [14], and [5]). It has also been shown that every connected Cayley graph has a cycle double cover [3].

2. Main results. The following lemma is crucial for the proofs of the main results. It is not hard to prove, and a proof can be found in [6], [11], [13] (see [8]).

LEMMA 2.1. *Let X be a cubic graph. The following two statements are equivalent.*

1. *The graph X admits a nowhere-zero 4-flow.*
2. *The graph X is 3-edge-colorable.*

The remainder of the paper addresses the following two results.

THEOREM 2.2. *Every cubic Cayley graph on a solvable group is 3-edge-colorable.*

R. Stong [12] proved that every Cayley graph $X(G;S)$ on a nilpotent group of even order has a 1-factorization as long as S is a minimal generating set for G. In particular, Stong's result implies that every cubic Cayley graph on a nilpotent group is 3-edge-colorable. The preceding theorem is an extension to solvable groups of this special case of Stong's theorem.

COROLLARY 2.3. *Every Cayley graph of degree at least two on a solvable group admits a nowhere-zero 4-flow.*

Proof. Let X be a Cayley graph of degree at least 2 on a solvable group. If X is of degree 2, then its components are cycles and it admits a nowhere-zero 2-flow. It is known that the edge connectivity of a connected Cayley graph is equal to its degree [10]. Thus, if X is of degree 4 or more, each component of X is 4-edge-connected and by Theorem 1.3 all of X admits a nowhere-zero 4-flow. This leaves only the case that X is cubic. In this case X is 3-edge-colorable by the preceding theorem. Lemma 2.1 then implies that X admits a nowhere-zero 4-flow and we are done. □

Proof of Theorem 2.2. Let $X = X(G;S)$ be a cubic Cayley graph on the solvable group G. The theorem is proved by induction on the order $|G|$ of G. We may assume that X is connected, for if it is not, we may apply the induction assumption to each component. R. Stong [12] has proved that every connected Cayley graph on a finite abelian group of even order has a 1-factorization (a partition of the edge set into 1-factors, that is, the chromatic index equals the degree). Thus, the result follows if G is abelian, and consequently, we assume that G is not abelian.

Let us examine S. We know that an element of order 2 in G generates a 1-factor of X. Since $|S| = 3$, we know that S contains either one element or three elements of order 2. In the latter case, X is 3-edge-colorable. Hence, we assume that $S = \{a, a^{-1}, b\}$, where $|b| = 2$ and $|a| = r > 2$.

If G has a nontrivial normal subgroup N such that $S \cap N = \emptyset$, then consider the quotient graph $\overline{X}$ obtained by first contracting every coset of N to a single vertex. If some vertex of a coset Ng is adjacent to d vertices of another coset Nh, then every vertex of Ng is adjacent to d vertices of Nh and vice versa because N is a normal subgroup. Then we put an edge of multiplicity d between the vertices corresponding to the cosets Ng and Nh in $\overline{X}$.

There are three possibilities for $\overline{X}$. First, $\overline{X}$ may be $3K_2$ (that is, two vertices joined by an edge of multiplicity 3). In this case, X is a bipartite graph. It is 3-edge-colorable because regular bipartite graphs have a 1-factorization.

Second, every vertex of $\overline{X}$ may be incident with an edge of multiplicity 1 and another edge of multiplicity 2. This means the quotient graph looks like an even length cycle in which every other edge around the cycle has multiplicity 2. Since each edge of $\overline{X}$ corresponds to a bipartite subgraph of X that is either regular of degree 1 or degree 2, it is again easy to see that X is 3-edge-colorable.

Third, $\overline{X}$ may be a cubic graph. Since G/N is also solvable, we know that $\overline{X}$ is 3-edge-colorable by induction. Each color class lifts to a 1-factor of X so that X is 3-edge-colorable too.

Thus we may assume that every nontrivial normal subgroup of G has nonempty intersection with S.

If the group $\langle a \rangle$ generated by a contains a nontrivial normal subgroup N, then $b \notin N$, as this would imply that $b \in \langle a \rangle$, that is, that G is abelian (cyclic). By the above assumption, $a \in N$, so $\langle a \rangle = N$. This implies that X itself is a generalized Petersen graph. F. Castagna and G. Prins [1] proved that all generalized Petersen graphs, other than the Petersen graph, are 3-edge-colorable. The Petersen graph is not a Cayley graph [4, p. 322]. Thus, we may assume that $\langle a \rangle$ contains no nontrivial normal subgroups of G.

We now use the previous assumption to reach two useful conclusions. If $ba^i = a^j b$ for some $i, j \in \{1, 2, \ldots, r-1\}$, then $\langle a^i, a^j \rangle$ is a normal subgroup of G contained in $\langle a \rangle$. By assumption we know this is not the case. Thus we conclude that

1. b does not commute with any a^i for $i = 1, 2, \ldots, r-1$ and
2. $ba^ib \notin \langle a \rangle$ for $i = 1, 2, \ldots, r-1$.

Since G is solvable, G contains a nontrivial abelian normal subgroup N (see [2, Prob. 11, p. 107]). We know that $S \cap N \neq \emptyset$ and $S \not\subseteq N$ (since X is connected). We consider the case that $a \in N$ and $b \notin N$. Then $[G : N] = 2$. Since N is abelian, $|a| = |bab| = r$ and by 2 above, $N = \langle a \rangle \times \langle bab \rangle$. Thus, $|N| = r^2$. Note that

$$N = \langle a \rangle \cup bab\langle a \rangle \cup ba^2b\langle a \rangle \cup \cdots \cup ba^{r-1}b\langle a \rangle$$

and

$$bN = b\langle a \rangle \cup ab\langle a \rangle \cup a^2b\langle a \rangle \cup \cdots \cup a^{r-1}b\langle a \rangle.$$

Denote the cycle of $ba^ib\langle a \rangle$ by

$$C_i = v_{i,0}v_{i,1} \ldots v_{i,r-1}v_{i,0},$$

where $v_{i,j} = ba^iba^j$ for $i, j \in \{0, 1, \ldots, r-1\}$, and denote the cycle of $a^ib\langle a \rangle$ by

$$D_i = u_{i,0}u_{i,1} \ldots u_{i,r-1}u_{i,0},$$

where $i, j \in \{0, 1, \ldots, r-1\}$. The b-edge incident with $v_{i,j} = ba^iba^j$ is also incident with $u_{j,i} = a^jba^i = ba^iba^jb$ because N is abelian and both ba^ib and a^j are in N. Let $P_i = C_i - v_{i,i}v_{i,i+1}$ and $Q_i = D_i - u_{i,i-1}u_{i,i}$ for $i = 0, 1, \ldots, r-1$ and with subscripts reduced modulo r. Then the union of all paths P_i and Q_i and the b-edges $v_{i,i}u_{i,i}, v_{i,i+1}u_{i+1,i}$, $i = 0, 1, \ldots, r-1$, is a Hamilton cycle of X. Thus, X is 3-edge-colorable.

We now consider the case that $b \in N$ and $a \notin N$. Observe that $\langle b \rangle \neq N$, for otherwise, $aba^{-1} \in N$ implies that $ab = ba$, which in turn implies that G is abelian. Now N is abelian and $b \in N$, so by 2 above, $a^k \notin N$ for any $k = 1, 2, \ldots, r-1$ and $a^i ba^{-i} \neq a^j ba^{-j}$ for $i \neq j$ (otherwise, a^{i-j} commutes with b).

Consider an auxiliary Cayley graph $X' = X(N; S')$ on N with $S' = \{b, aba^{-1}\}$. Both elements have order 2 and define edges in a Cayley graph on an abelian group. Thus, X' consists of vertex-disjoint 4-cycles. A typical 4-cycle has the form y, yb, $ybaba^{-1}$, $ybaba^{-1}b, y$. Back in the original graph X, each vertex z of a 4-cycle corresponds to the r-cycle $z, za, za^2, \ldots, za^{r-1}$. Notice that there is an edge joining yba and $ybaba^{-1}a = ybab$ and an edge joining ya and $ybaba^{-1}ba = yab$. Hence, the typical 4-cycle mentioned above lifts to a $4r$-cycle in X by removing the edges (y, ya), (yb, yba), $(ybaba^{-1}, ybaba^{-1}a)$, and $(ybaba^{-1}b, ybaba^{-1}ba)$ from the four r-cycles corresponding to the vertices of the 4-cycle and by replacing them with the four edges (y, yb), $(ybaba^{-1}, ybaba^{-1}b)$, $(ya, ybaba^{-1}ba)$, and $(yba, ybaba^{-1}a)$. Hence, X has a 2-factor made up of cycles of length $4r$, so X is 3-edge-colorable. This completes the proof of the theorem. □

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