SIAM J. DISC. MATH.
Vol. 3, No. 3, pp. 431–438, August 1990

012

FINDING CRITICAL INDEPENDENT SETS AND CRITICAL VERTEX SUBSETS ARE POLYNOMIAL PROBLEMS*

CUN-QUAN ZHANG†

Abstract. An independent set J_c of a graph G is called *critical* if

$$|J_c| - |N(J_c)| = \max\{|J| - |N(J)| : J \text{ is an independent set of } G\},$$

and a vertex subset U_c is called *critical* if

$$|U_c| - |N(U_c)| = \max\{|U| - |N(U)| : U \text{ is a vertex subset of } G\}.$$

In this paper, it will be shown that finding a critical independent set and a critical vertex subset of a graph are solvable in polynomial time.

Key words. independent set, polynomial algorithm

AMS(MOS) subject classifications. 05C35, 68R10

1. Introduction. It has been proved by mathematicians that finding a maximum independent set in certain kinds of graphs is solvable in polynomial time (for example, line graphs, bipartite graphs, circle graphs, circular arc graphs, and claw free graphs (see [GJ])), but it is well known that it is an NP-complete problem for general graphs (see [GJS]). In this paper, we will investigate another problem—finding a certain kind of independent set in general graphs. An independent set J_c of a graph G is called critical if $|J_c| - |N(J_c)|$ is the maximum of $|J| - |N(J)|$ over all independent sets J of G, where $N(J)$ is the set of all vertices of G adjacent to some vertex of J. It will be proved in this paper that finding a critical independent set of a graph is solvable in polynomial time. Let

$$\alpha_c = \max\{|J| - |N(J)| : J \text{ is an independent set of } G\},$$

which is a parameter of a graph G and is called the *critical independence number* of G. The critical independence number α_c of a graph plays the central role in the study of fractional independence functions and fractional matching functions of graphs [GZ]. (It is proved in [GZ] that the fractional independence number and the fractional matching number of a graph G are $(n - \alpha_c)/2$ and $(n + \alpha_c)/2$, respectively, where $n = |V(G)|$.)

Some related problems and parameters of graphs will also be investigated in this paper. A vertex subset U_c of a graph $G = (V, E)$ is called *critical* if $|U_c| - |N(U_c)|$ is the maximum of $|U| - |N(U)|$ over all vertex subsets U of G. Let

$$\mu_c = \max\{|U| - |N(U)| : U \text{ is a vertex subset of } G\}$$

which is a parameter of a graph G. Some similar parameters of graphs have been studied by Woodall [WD] and Mohar [MB]. The binding number $b(G)$ [WD] and the isoperimetric number $i(G)$ [MB] of a graph G are defined as the following:

$$b(G) = \min\left\{\frac{|N(U)|}{|U|} : U \subset V(G), U \neq \emptyset \text{ and } N(U) \neq V(G)\right\};$$

$$i(G) = \min\left\{\frac{|\partial(U)|}{|U|} : U \subset V(G), U \neq \emptyset \text{ and } |U| \leq \frac{|V(G)|}{2}\right\}$$

* Received by the editors January 16, 1989; accepted for publication (in revised form) September 20, 1989.

† Department of Mathematics, West Virginia University, Morgantown, West Virginia 26506. This research was partially supported by Air Force Office of Scientific Research grant 89-0068.

(where $\partial(U)$ is the number of edges of G with one end vertex in U). Later in this paper, it will be proved that $\alpha_c = \mu_c$.

Since the empty set is an independent set, and the set of all vertices of a graph G is also a vertex subset of G, it is trivial that

$$\alpha_c \geqq 0 \quad \text{and} \quad \mu_c \geqq 0$$

for any graph G. Note that the empty set and the entire graph are critical vertex subsets of some connected graph G if $\mu_c(G) = 0$. If we are to avoid these two trivial vertex subsets $\emptyset$ and $V(G)$, we may consider the following parameter of a graph G:

$$\mu_c' = \max \{|U| - |N(U)| : U \subset V(G), U \neq \emptyset \text{ and } U \neq V(G)\}.$$

But for any connected graph G and a vertex v of G, we have that $N(V(G)\setminus\{v\}) = V(G)$, and therefore the parameter $\mu_c'(G)$ still has a lower bound -1. In order to get more information about graphs, we prefer to consider only those vertex subsets U of a graph G such that $U \neq \emptyset$ and $N(U) \neq V(G)$ which is similar to the definition of the binding number of graphs. A vertex subset U of G is called *proper* if $U \neq \emptyset$ and $N(U) \neq V(G)$. A proper vertex subset U_{pc} of G is called *critical* if $|U_{pc}| - |N(U_{pc})|$ is the maximum of $|U| - |N(U)|$ over all proper vertex subsets U of G. The parameter μ_{pc} of a graph G is defined as the following:

$$\mu_{pc} = \max \{|U| - |N(U)| : U \subset V(G), U \neq \emptyset \text{ and } U \neq V(G)\}.$$

The problems that will be proved to be solvable in polynomial time are listed as the following.

INSTANCE. Let $G = (V, E)$ be a graph with the vertex set V and the edge set E and k be an integer.

PROBLEM 1. Is there an independent set J of G such that

$$|J| - |N(J)| \geqq k?$$

PROBLEM 1*. Find a critical independent set J_c and the critical independence number α_c of G. (That is, to find

$$\alpha_c = |J_c| - |N(J_c)| = \max \{|J| - |N(J)| : J \text{ is an independent set of } G\}).$$

PROBLEM 2. Is there a vertex set U of G such that

$$|U| - |N(U)| \geqq k?$$

PROBLEM 2*. Find a critical vertex subset U_c and the parameter μ_c of G. (That is, to find

$$\mu_c = |U_c| - |N(U_c)| = \max \{|U| - |N(U)| : U \text{ is a vertex subset of } G\}).$$

PROBLEM 3. Is there a proper vertex subset U of G such that

$$|U| - |N(U)| \geqq k?$$

PROBLEM 3*. Find a critical proper vertex subset U_{pc} and the parameter μ_{pc} of G. (That is, to find

$$\mu_{pc} = |U_{pc}| - |N(U_{pc})| = \max \{|U| - |N(U)| : U \subset V(G), U \neq \emptyset \text{ and } U \neq V(G)\}).$$

2. Main results.

THEOREM 1. *Problems* 3 *and* 3* *are solvable in polynomial time.*

Before we prove Theorem 1 we would like to consider the following problems first. Theorem 1 will be a corollary of Theorem 2.

INSTANCE. Let $G = (V, E)$ be a graph with the vertex set V and the edge set E, $\{u, v\}$ be an ordered pair of nonadjacent vertices of G, and k be an integer.

PROBLEM 4. Is there a vertex subset U of G such that

$$u \in U, v \notin N(U) \text{ and}$$
$$|U| - |N(U)| \geqq k?$$

PROBLEM 4*. Find a vertex subset U_o of G such that

$$|U_o| - |N(U_o)| = \max\{|U| - |N(U)| : U \subset V(G), u \in U \text{ and } v \notin N(U)\}.$$

The vertex subset U_o found in problem 4* is called (u, v)-critical subset of G.

THEOREM 2. *Problems* 4 *and* 4* *are solvable in polynomial time.*

The following lemmas will be used in the proof of Theorem 2.

LEMMA 3 (Hall's Theorem [HP]). *Let $B = (V_1, V_2; E)$ be a bipartite graph. The graph B has a matching covering all vertices of V_2 if and only if $|U| \leqq |N(U)|$ for any subset U of V_2.*

LEMMA 4. *Let $B = (V_1, V_2; E)$ be a bipartite graph. Assume that there is no matching of B covering all vertices of V_2. We will have the following conclusions:*

(i) *There is a subset U of V_2 such that $|N(U)| < |U|$;*

(ii) *Let U_0 be a subset of V_2 such that $|U_0| - |N(U_0)|$ is as great as possible, then there is a matching of the induced bipartite subgraph $(U_0, (N(U_0); E[U_0, N(U_0)])$ covering all vertices of $N(U_0)$.*

Note that if A and B are a pair of disjoint vertex subsets of a graph G, the set of all edges joining A and B is denoted by $E(A, B)$.

Proof. The conclusion of (i) is an immediate corollary of Hall's Theorem.

Let U be a subset of V_2 such that $|U| - |N(U)|$ is as great as possible. Let $B' = (U, N(U); E[U, N(U)])$ be the subgraph of B induced by $U \cup N(U)$. We claim that $|X| \leqq |N(X) \cap U|$ for any subset X of $N(U)$. If not, let $X \subseteq N(U)$ such that

$$|X| > |N(X) \cap U|.$$

We will consider the subset $Y = U \backslash N(X)$. Note that

$$N(Y) = N(U \backslash N(X)) \subseteq N(U) \backslash X,$$

and

$$\begin{aligned} |Y| - |N(Y)| \geqq |U \backslash N(X)| - |N(U) \backslash X| &= [|U| - |U \cap N(X)|] - [|N(U)| - |X|] \\ &= [|U| - |N(U)|] + [|X| - |U \cap N(X)|] \\ &> |U| - |N(U)|. \end{aligned}$$

This contradicts the choice of U that $|U| - |N(U)|$ is maximum. So by Hall's Theorem, there is a matching in B' which covers all vertices of $N(U)$. □

Proof of Theorem 2. We only prove that Problem 4* is solvable in polynomial time. Let $G = (V, E)$ be a graph with the vertex set $V = \{1, 2, 3, \cdots, n\}$ and the edge set E. We will consider the ordered pair of vertices $(1, 2)$ of G and find a $(1, 2)$-critical vertex subset.

Define a bipartite graph $B = (X, Y; E_B)$ where

$$X = \{x_1, \cdots, x_n\},$$
$$Y = \{y_1, \cdots, y_n\}, \quad \text{and}$$
$$E_B = \{(x_i, y_j) : (i, j) \text{ is an edge of } G\}.$$

Let V' be a subset of $V(G)$, then the corresponding subsets in X and Y are denoted by $X(V')$ and $Y(V')$, respectively. For example, if $V' = \{i_1, \cdots, i_t\}$, then

$$X(V') = X(\{i_1, \cdots, i_t\}) = \{x_{i_1}, \cdots, x_{i_t}\}$$

and $Y(V') = Y(\{i_1, \cdots, i_t\}) = \{y_{i_1}, \cdots, y_{i_t}\}$. (Here X and Y can be considered bijections mapping $\{1, 2, \cdots, n\}$ onto $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_n\}$). If $W = \{x_{i_1}, \cdots, x_{i_t}\} \subset X$ (or $W = \{y_{i_1}, \cdots, y_{i_t}\} \subset Y$), then the corresponding subset $\{i_1, \cdots, i_t\}$ of $V(G)$ is denoted by $X^{-1}(W)$ (or $Y^{-1}(W)$, respectively). The set of all neighbors of a vertex u in B is denoted by $N_B(u)$. If i is a vertex of G, then

$$N_B(x_i) = \{y_j \in Y : (x_i, y_j) \in E_B\} = \{y_j \in Y : (i, j) \in E(G)\} = Y(N(i)).$$

A weight $w: X \cup Y \to [0, 2]$ is called a $(1, 2)$-*proper weight* of B if

$$w(x_1) = 2,$$

$$w(y_2) = 1,$$

$$1 \leqq w(x_i) \leqq 2 \quad \text{for each vertex } x_i \in X,$$

$$0 \leqq w(y_i) \leqq 1 \quad \text{for each vertex } y_i \in Y$$

$$\text{and } 0 \leqq w(x_i) + w(y_i) \leqq 2 \quad \text{for each edge } (x_i, y_i) \in E_B.$$

The *total weight* $\sum_{u \in X \cup Y} w(u)$ of B is denoted by $w(B)$. A $(1, 2)$-proper weight w_m of B is called *optimum* if

$$w_m(B) = \max\{w(B) : w \text{ is a } (1,2)\text{-proper weight of } B\}.$$

It is obvious that finding an optimum $(1, 2)$-proper weight of B is a linear programming problem. Hence it is solvable in polynomial time. The purpose of the investigation of an optimum $(1, 2)$-proper weight w_m of B is to prove that the vertex subset $\{i \in V(G) : w_m(x_i) > 1\}$ is a $(1, 2)$-critical subset of G.

I. Let w_m be an optimum $(1, 2)$-proper weight of B and U_o be a $(1, 2)$-critical vertex subset of G. We claim that

$$(1) \qquad w_m(B) \geqq 2n + |U_o| - |N(U_o)| = 2n + \alpha_c.$$

Consider the following weight w_1 of B:

$$w_1(x_i) = \begin{cases} 2 & \text{if } i \in U_o \\ 1 & \text{otherwise} \end{cases}$$

$$\text{and } w_1(y_j) = \begin{cases} 0 & \text{if } j \in N(U_o) \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that w_1 is a $(1, 2)$-proper weight of B and

$$w_1(B) = 2|U_o| + (n - |U_o|) + (n - |N(U_o)|) = 2n + |U_o| - |N(U_o)|.$$

By the choice of w_m, we have verified the inequality (1).

II. Let

$$X_b = \{x_i : w_m(x_i) > 1\},$$

$$Y_s = \{y_i : w_m(y_i) < 1\},$$

$$X_{b'} = \{x_i : 1 < w_m(x_i) < 2\}, \qquad \text{and}$$

$$Y_{s'} = \{y_i : 0 < w_m(y_i) < 1\}.$$

By the definition of (1, 2)-proper weight, it is obvious that $N_B(X_b) \subseteq Y_s$ and $N_B(Y_{s'}) \subseteq [X \backslash X_b] \cup X_{b'}$.

III. *Case* 1. Suppose there is a matching M in the induced subgraph $B(X_{b'} \cup Y_{s'})$ covering all vertices of $Y_{s'}$. We claim that X_b is a (1, 2)-critical vertex subset in this case. We are to adjust the weight w_m so that the new weight of each vertex in $X_{b'}$ is two and the new weight of each vertex in $Y_{s'}$ is zero. If we can verify that this new weight is optimum, by the inequality (1), it can be shown that $X^{-1}(X_b)$ is a (1, 2)-critical vertex set of G.

If (u, v) is an edge of M, then let $u = M(v)$ and $v = M(u)$. The sets of vertices of $X_{b'}$ covered and not covered by M, are denoted by $M(Y_{s'})$ and $X_{b'} \backslash M$, respectively. Thus $X_{b'} \cup Y_{s'} = Y_{s'} \cup M(Y_{s'}) \cup (X_{b'} \backslash M)$ since M covers all vertices of $Y_{s'}$.

Consider the following weight w_2 of B:

$$w_2(v) = \begin{cases} 2 & \text{if } v \in X_{b'} \\ 0 & \text{if } v \in Y_{s'} \\ w_m(v) & \text{otherwise.} \end{cases}$$

Since any vertex of Y adjacent to a vertex x_i of $X_{b'}$ must be in Y_s, in which the weight of each vertex is zero, w_2 is a (1, 2)-proper weight of B. Note that w_m is optimum and the total weight $w_m(B)$ cannot be less than $w_2(B)$. We claim that w_2 is also an optimum (1, 2)-proper weight of B by proving that $w_m(B) \leqq w_2(B)$. Since

$$\{v \in X \cup Y : w_m(v) \neq w_2(v)\} = X_{b'} \cup Y_{s'},$$

we must have that

$$\begin{aligned} w_m(B) - w_2(B) &= \sum_{v \in X_{b'}} [w_m(v) - w_2(v)] + \sum_{v \in Y_{s'}} [w_m(v) - w_2(v)] \\ &= \sum_{v \in Y_{s'}} \{[w_m(v) - w_2(v)] + [w_m(M(v)) - w_2(M(v))]\} \\ &\quad + \sum_{v \in X_{b'} \backslash M} [w_m(v) - w_2(v)]. \end{aligned}$$

Here

$$\begin{aligned} &[w_m(v) - w_2(v)] + [w_m(M(v)) - w_2(M(v))] \\ &\qquad = [w_m(v) + w_m(M(v))] - [w_2(v) + w_2(M(v))] \leqq 2 - (0 + 2) = 0 \end{aligned}$$

for any $v \in Y_{s'}$, and

$$w_m(v) - w_2(v) < 2 - 2 = 0$$

for any $v \in X_{b'} \backslash M$. This implies that $w_m(B) - w_2(B) \leqq 0$. Therefore w_2 is also an optimum (1, 2)-proper weight of B and $w_m(B) = w_2(B)$. The total weight of w_2 is

$$w_2(B) = 2|X_b| + |X \backslash X_b| + |Y \backslash Y_s|.$$

Since $N_B(X_b) \subseteq Y_s$, we must have that

$$\begin{aligned} w_m(B) = w_2(B) &\leqq 2|X_b| + (n - |X_b|) + (n - |N(X_b)|) = 2n + |X_b| - |N(X_b)| \\ &= 2n + |X^{-1}(X_b)| - |N(X^{-1}(X_b))| \leqq 2n + \alpha_c (\text{by the definition of } \alpha_c). \end{aligned}$$

By (1), all equalities hold, and therefore $X^{-1}(X_b) = \{i \in V(G) : w_m(x_i) > 1\}$ is a (1, 2)-critical vertex subset of G.

IV. *Case* 2. If there is no matching of $B(X_{b'} \cup Y_{s'})$ covering all vertices of $Y_{s'}$, by Lemma 4, there is a subset Y_o of $Y_{s'}$ such that: (i) $|Y_o| > |N_B(Y_o) \cap X_{b'}|$ and, (ii) there is a matching M' in the induced bipartite subgraph $B(Y_o \cup [N_B(Y_o) \cap X_{b'}])$ covering all vertices of $N_B(Y_o) \cap X_{b'}$. We are to adjust the weight w_m so that the new weight of each vertex in $Y_o \cup [N_B(Y_o) \cap X_{b'}]$ is 1. We will find that the new weight is greater than w_m. It will contradict that w_m is optimum.

Consider the following weight w_3:

$$w_3(v) = \begin{cases} 1 & \text{if } v \in Y_o \cup [N_B(Y_o) \cap X_{b'}] \\ w_m(v) & \text{otherwise.} \end{cases}$$

The weight w_3 is (1, 2)-proper since $x_1 \notin N(Y_{s'})$ and any vertex adjacent to a vertex of Y_o must be in $[X \backslash X_b] \cup [N_B(Y_o) \cap X_{b'}]$ in which the weight of each vertex is one. Note that w_m is an optimum (1, 2)-proper weight of B, thus the total weight of w_m cannot be less than the total weight of w_3. Since

$$\{v \in V(B) : w_m(v) \neq w_3(v)\} \subseteq Y_o \cup [N_B(Y_o) \cap X_{b'}]$$

$$= [N_B(Y_o) \cap X_{b'}] \cup M'[N_B(Y_o) \cap X_{b'}] \cup [Y_o \backslash M'],$$

where $M'[N_B(Y_o) \cap X_{b'}]$ and $[Y_o \backslash M']$ are the sets of vertices of Y_o covered and not covered by M', respectively, we must have that

$$\begin{aligned} w_m(B) - w_3(B) &= \sum_{v \in N_B(Y_o) \cap X_{b'}} [w_m(v) - w_3(v)] + \sum_{v \in Y_o} [w_m(v) - w_3(v)] \\ &= \sum_{v \in N_B(Y_o) \cap X_{b'}} \{[w_m(v) - w_3(v)] + [w_m(M(v)) - w_3(M(v))]\} \\ &\quad + \sum_{v \in Y_o \backslash M'} [w_m(v) - w_3(v)]. \end{aligned}$$

But

$$\begin{aligned} &[w_m(v) - w_3(v)] + [w_m(M(v)) - w_3(M(v))] \\ &\qquad = [w_m(v) + w_m(M(v))] - [w_3(v) + w_3(M(v))] \leqq 2 - (1+1) = 0 \end{aligned}$$

for any $v \in N_B(Y_o) \cap X_{b'}$, and

$$w_m(v) - w_3(v) < 1 - 1 = 0$$

for any $v \in Y_o \backslash M'$. Since $|Y_o| > |N_B(Y_o) \cap X_{b'}|$, the set $Y_o \backslash M'$ is not empty and we have that $w_m(B) - w_3(B) < 0$. This contradicts that w_m is optimum and completes the proof of the theorem. □

Proof of Theorem 1. For any ordered pair of nonadjacent vertices $\{u, v\}$ of G, by Theorem 2, we can find a (u, v)-critical subset in polynomial time. The (u, v)-critical subset is denoted by $U_{C(u,v)}$. Then choose a (u_o, v_o)-critical vertex subset $U_{C(u_o,v_o)}$ such that

$$|U_{C(u_o,v_o)}| - |N(U_{C(u_o,v_o)})| = \max \{|U_{C(u,v)}| - |N(U_{C(u,v)})| : (u,v) \text{ are an ordered pair of nonadjacent vertices of } G\}$$

which is a proper critical subset of G desired in Problem 3*. The total cost of finding a critical proper vertex subset is polynomial since the cost of finding a (u, v)-critical vertex subset is polynomial and the number of pairs of nonadjacent vertices in G is at most $\binom{n}{2}$. □

THEOREM 5. *Problems* 2 *and* 2* *are solvable in polynomial time.*

Proof. Let $G = (V, E)$ be a graph. Consider a new graph G' by adding two isolated vertices x, y, to G. Let U_c be an (x, y)-critical subset of G'. Obviously, $x, y \in U_c$ and it is clear that $U_c \setminus \{x, y\}$ is a critical subset of G.

Alternating proof of the theorem. Define a bipartite graph $B = (X, Y; E_B)$ where $X = \{x_1, \cdots, x_n\}$, $Y = \{y_1, \cdots, y_n\}$ and $E_B = \{(x_i, y_j) : (i, j) \text{ is an edge of } G\}$. Assign a weight w to the vertex set of B such that $w: X \cup Y \to [0, 2]$ and

$$1 \leqq w(x_i) \leqq 2 \qquad \text{for each vertex } x_i \in X,$$

$$0 \leqq w(y_i) \leqq 1 \qquad \text{for each vertex } y_i \in Y$$

$$\text{and } 0 \leqq w(x_i) + w(y_i) \leqq 2 \quad \text{for each edge } (x_i, y_i) \in E_B.$$

The total weight $\sum_{u \in X \cup Y} w(u)$ is denoted by $w(B)$. Let

$$w_m(B) = \max \{ w(B) : w \text{ is a weight of } B \text{ satisfying the above definition} \}.$$

By an argument similar to the proof of Theorem 2, we can prove that the set $V_b = \{i \in V(G) : w_m(x_i) > 1\}$ is a critical vertex subset of G. □

THEOREM 6. *Problems* 1 *and* 1* *are solvable in polynomial time.*

Before the proof of Theorem 6, we will prove a Theorem by which Theorem 6 is an immediate corollary.

THEOREM 7. *Let* $G = (V, E)$ *be a graph.*

(i) *Let* U_c *be a critical vertex subset of* G *and* $T_1, \cdots, T_h$ *be all nontrivial components of the induced subgraph* $G(U_c)$. *Then* $J = V(U_c) \setminus [V(T_1) \cup \cdots \cup V(T_h)]$ *is a critical independent set of* G *and* $|J| - |N(J)| = |U_c| - |N(U_c)|$.

(ii) $\max \{|J| - |N(J)| : J$ *is an independent set of* $G\} = \max \{|U| - |N(U)| : U$ *is a vertex subset of* $G\}$. *That is, any critical independent set is also a critical vertex subset of* G *and therefore* $\alpha_c = \mu_c$.

Proof. It is obvious that

$$(2) \qquad 0 \leqq \alpha_c = \max \{|J| - |N(J)| : J \text{ is an independent set of } G\}$$
$$\leqq \max \{|U| - |N(U)| : U \text{ is a vertex subset of } G\} = \mu_c \cdots$$

since any independent set is a vertex subset of G.

Let U_c be a critical vertex subset of G. The theorem is trivial if U_c is an empty set. *Assume that* U_c *is a counterexample to the theorem containing a minimum number of vertices.* By this assumption, U_c cannot be an independent set of G. Let T be a nontrivial component of the induced subgraph $G(U_c)$. It is clear that $V(T) \subseteq N(T)$ since T is not a singleton. Thus

$$|U_c \setminus T| - |N(U_c \setminus T)| \geqq [|U_c| - |T|] - [|N(U_c)| - |T|] = |U_c| - |N(U_c)|.$$

It implies that $U_c \setminus T$ is also a critical vertex subset of G which contains less vertices than U_c. It contradicts the choice of U_c and therefore completes the proof of the theorem. □

Proof of Theorem 6. Let U_c be a critical vertex subset of G. Let $T_1, \cdots, T_t$ be all nontrivial components of the induced subgraph $G(U_c)$. Then by Theorem 7, $U_c \setminus \{T_1, \cdots, T_t\}$ is a critical independent set of G. Since finding a critical vertex subset U_c and deleting all vertices of U_c incident with some edge of $G(U_c)$ need only polynomial cost, finding a critical independent set is solvable in polynomial time.

Alternating proof of the theorem (also see [GZ]). Consider a weight $w: V(G) \to [0, 1]$ such that

$$0 \leqq w(v) \leqq 1 \qquad \text{for each vertex } v \text{ of } G$$

$$\text{and } w(u) + w(v) \leqq 1 \quad \text{for each edge } (u, v) \text{ of } G.$$

The total weight $\sum_{u\in V(G)} w(u)$ is denoted by $w(G)$. Let

$$w_m(G) = \max\{w(G): w \text{ is a weight of } B \text{ satisfies the above definition}\}.$$

Obviously finding w_m is a linear programing problem. By an argument similar to the proof of Theorem 2, we can prove that the set $V_b = \{v \in V(G): w_m(v) > \frac{1}{2}\}$ is a critical independent set of G. $\square$

(Note that the weight w defined above is called a fractional independence function of a graph G which was introduced by Domke, Hedetniemi, and Laskar in [DHL] and by Grinstead and Slater in [GS2], and was studied by Grinstead and Slater in [GS1] and by Goldwasser and Zhang in [GZ].)

REFERENCES

[DHL] G. S. Domke, S. T. Hedetniemi, and R. C. Laskar, *Fractional packings, coverings, and irredundance in graphs*, preprint.

[GJ] M. R. Garey and D. S. Johnson, *Computers and Intractability*, W. H. Freeman and Company, New York, 1979.

[GJS] M. R. Garey, D. S. Johnson, and L. Stockmeyer, *Some simplified* NP-*complete graph problems*, Theoret. Comput. Sci., 1 (1976), pp. 237–267.

[GS1] D. L. Grinstead and P. J. Slater, *On fractional covering and fractional independence parameters*, Third Carbondale Combinatorics Conference, Southern Illinois Univ., Carbondale, IL, 1988.

[GS2] ———, *Fractional domination and fractional packings in graphs*, preprint.

[GZ] J. Goldwasser and C. Q. Zhang, *Fractional independence number and fractional matching number*, preprint.

[HP] P. Hall, *On representatives of subsets*, J. London Math. Soc., 10 (1935), pp. 26–30.

[MB] B. Mohar, *Isoperimetric number of graphs*, preprint.

[MG] G. J. Minty, *On maximal independent sets of vertices in claw-free graphs*, J. Combin. Theory Ser. B, 28 (1977), pp. 283–304.

[WD] D. R. Woodall, *The binding number of a graph and its Anderson number*, J. Combin. Theory Ser. B, 15 (1973), pp. 225–255.