Strong Circuit Double Cover of Some Cubic Graphs

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Abstract: Let *C* be a given circuit of a bridgeless cubic graph *G*. It was conjectured by Seymour that *G* has a circuit double cover (CDC) containing the given circuit *C*. This conjecture (strong CDC [SCDC] conjecture) has been verified by Fleischner and Häggkvist for various families of graphs and circuits. In this article, some of these earlier results have been improved:

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(1) if $H = G - C$ contains a Hamilton path or a Y-tree of order less than 14, then G has a CDC containing C; (2) if $H = G - C$ is connected and $|V(H)| \leq 6$, then G has a CDC containing C. J. Graph Theory 78: 131–142, 2015

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1. INTRODUCTION

Most notation and terminology not defined in this article can be found in standard textbooks on graph theory, for instance [3], [33], and [35]. All graphs we consider in this article may have multiple edges but no loops. A *circuit* is a connected 2-regular subgraph.

The circuit double cover (CDC) conjecture has been recognized as one of the major open problems in graph theory.

Conjecture 1.1 (CDC conjecture, [31], [28], [20], and [27]). *Every bridgeless graph has a family of circuits that covers every edge precisely twice.*

As pointed out in [22], it is sufficient to consider cubic graphs only for the CDC problem since a smallest counterexample to the conjecture is cubic (by applying vertex splitting method).

Some stronger versions of the CDC conjecture have been proposed (such as [1], [2], [5], [7], [21], [22], [23], [24], [26], [27], [32], etc.) The following open problem is one of the most well-known conjectures in this subject.

Conjecture 1.2 (Strong circuit double cover (SCDC) conjecture, Seymour, see [8], p. 237, and [9]). *Let G be a bridgeless cubic graph and C be any given circuit in G, then the graph G has a CDC $\mathcal{F}$ containing C.*

The SCDC conjecture (Conjecture 1.2) has been verified for various families of graphs, such as 3-edge-colorable cubic graphs [27], snarks of order at most 36 [4], a circuit C of length at least $|V(G)| - 1$ [10], and some special families of graphs with given circuits described in [12], [15] (see Theorems 1.4 and 1.7), etc.

Note that the SCDC conjecture is not true if the given circuit C is replaced with a family of edge-disjoint circuits (the Petersen graph is a counterexample).

The CDC conjecture has been verified by Tarsi for graphs with Hamilton paths.

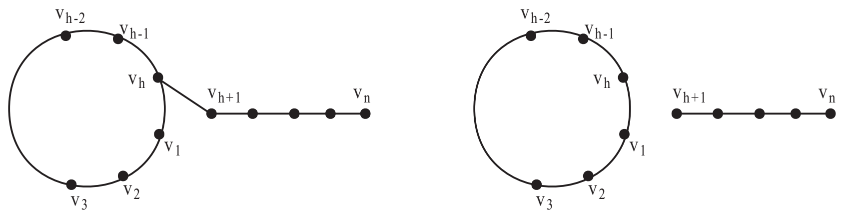
Theorem 1.3 (Tarsi [29]). *Every bridgeless cubic graph containing a Hamilton path has a CDC.*

Theorem 1.3 is further strengthened in [19] for oddness 2 graphs and also strengthened in [15] with respect to Conjecture 1.2 (the SCDC conjecture).

Theorem 1.4 (Fleischner and Häggkvist [15]). *Let G be a bridgeless cubic graph with a Hamilton path $v_1, \ldots, v_n$ and $v_1 v_h \in E(G)$ $(h > 2)$. Then, G has a CDC $\mathcal{F}$ that contains the circuit $v_1, \ldots, v_h v_1$.*

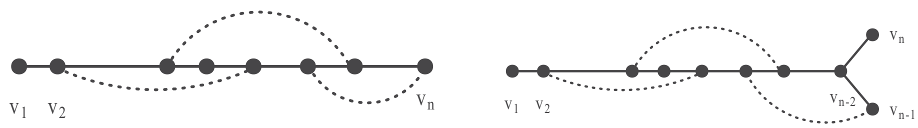
In this article, we are interested in extending Theorems 1.3 and 1.4 as follows.

Problem 1.5. *Let G be a bridgeless cubic graph with a given circuit C. If $G - V(C)$ contains a Hamilton path P, can we find a CDC $\mathcal{F}$ of G that contains the circuit C?*



(a) Theorems 1.4 (b) Problem 1.5

FIGURE 1. A Hamilton path $v_1v_2, \ldots, v_hv_{h+1}, \ldots, v_n$ can be found in left figure. There is no Hamilton path in right figure, i.e., two end vertices v_{h+1} and v_n are not adjacent to the circuit $v_1v_2 \ldots v_h$.



(a) Hamilton Path (b) Y-tree

FIGURE 2. Hamilton Path and Y-tree (solid line) spanning in H.

Or, a more general question as following.

Problem 1.6. *Let G be a bridgeless cubic graph with a given circuit C. If $G - V(C)$ is connected, can we find a CDC $\mathcal{F}$ of G that contains the circuit C?*

For Problem 1.6, Fleischner and Häggkvist has the following result.

Theorem 1.7 (Fleischner and Häggkvist [12]). *Let G be a bridgeless cubic graph with a given circuit C. If $G - V(C)$ is connected and of order at most 4, then G has a CDC $\mathcal{F}$ that contains the circuit C.*

Note that the difference between Theorem 1.4 and Problem 1.5 is *whether there is an edge joining an endvertex of P and some vertex of C*. If yes, the lollipop method (Section 2) is applied and Theorem 1.4 follows [15]. However, if the circuit C and path P are not connected in such way, more structural studies are necessary beyond the application of the lollipop method (see Fig. 1).

In this article, we obtain some partial results (Theorems 1.10 and 1.11) related to both problems that strengthen some of the results by Fleischner and Häggkvist.

Almost all results in this article are presented for cubic graphs only. However, they can all be converted to results for general graphs by applying vertex-splitting methods [6].

For the sake of convenience, we denote by (G, C) a pair consisting of a cubic graph G and a given circuit C of G.

Definition 1.8. *Let G be a graph with $\Delta(G) \geq 3$. The suppressed graph of G is the graph obtained from G by replacing each maximal subdivided edge with a single edge, and is denoted by G^s.*

Definition 1.9. *A spanning tree T of the graph H is called a spanning Y-tree if T consists of a path $v_1, \ldots, v_{t-1}$ and $v_{t-2}v_t \in E(T)$ (see Fig. 2).*

The following are the main results of the article.

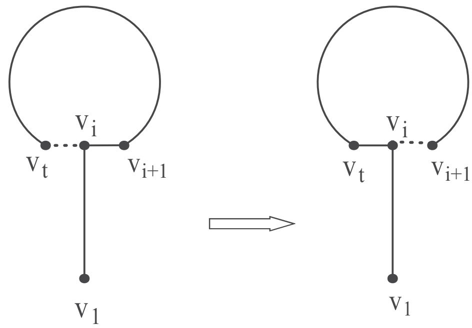


FIGURE 3. Lollipop detour $P \Rightarrow P'$.

Theorem 1.10. *Let C be a given circuit of a bridgeless cubic graph G. If $H = G - C$ contains a Hamilton path or a Y-tree of order ≤ 14, then G has a CDC containing C.*

Theorem 1.11. *Let C be a given circuit of a bridgeless cubic graph G. If $H = G - C$ is connected and of order ≤ 6, then G has a CDC containing C.*

2. LOLLIPOP METHOD AND ITS APPLICATIONS

Definition 2.1. *Let $P = v_1v_2, \ldots, v_t$ be a path of a cubic graph. Let $v_i \in N(v_t) \cap \{v_2, v_3, \ldots, v_{t-1}\}$. The subgraph $P' = v_1v_2, \ldots, v_iv_t, \ldots, v_{i+1}$ is a path obtained from P via a lollipop detour (see Fig. 3).*

The following lemma will be proved by the *lollipop method*, a technique that was first introduced by Thomason [30].

Lemma 2.2. *Let G be a cubic graph of order n and $C = v_1v_2, \ldots, v_rv_1$ be a circuit of G. Then*

(1) *either there is another circuit $C' = v_1v_2, \ldots, v_1$ containing the edge v_1v_2 with $V(C) = V(C')$ and $E(C) \neq E(C')$;*
(2) *or there is a path $P = v_1v_2, \ldots, z$ starting at the vertex v_1 and edge v_1v_2, and $V(P) = V(C) \cup \{z\}$ for some vertex $z \notin V(C)$.*

Proof. Construct an auxiliary graph A_G. Each vertex of A_G is a path P of G starting at the vertex v_1 and edge v_1v_2 with $V(P) = V(C)$, and P_1 is adjacent to P_2 if and only if P_1 is obtained from P_2 via a lollipop detour. Therefore, every vertex in A_G has degree 2 or 1.

Note that $P = v_1v_2, \ldots, v_r$ is a degree-1 vertex in the auxiliary graph A_G. Since the component of A_G containing the vertex P is a path, it must have another degree-1 vertex $P' = v_1v_2, \ldots, x$. The case $v_1 \in N(x)$ implies that P' can be extended to a distinct circuit C', and otherwise $N(x)$ contains a new vertex z not in $V(C)$, as we desired. ■

Definition 2.3. *Let H be a graph of order t with $\Delta \leq 3$. A Hamilton path $T = v_1, \ldots, v_t$ or a Y-tree $T = v_1, \ldots, v_{t-1} + v_{t-2}v_t$ is small ended if $d_H(v_1) \leq 2$.*

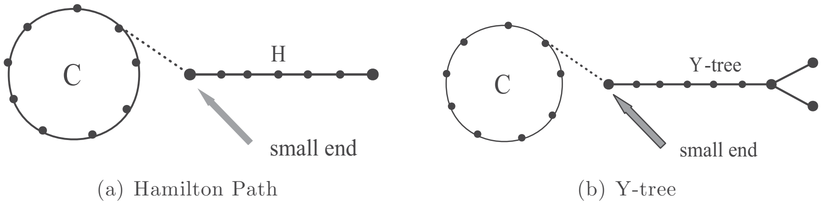


FIGURE 4. Small-ended Hamilton path and small-ended Y-tree.

In Figure 4, a small-ended Hamilton path and a small-ended spanning Y-tree are illustrated in $G - V(C)$.

Here, Theorem 1.4 is extended as follows, which not only includes the proof of Theorem 1.4 but also a result for small-ended Y-trees.

Theorem 2.4. *For a pair (G, C), if $H = G - V(C)$ has either a small-ended Hamilton path $P_0 = x_1, \ldots, x_t$ with $d_H(x_1) \leq 2$, or a small-ended Y-tree consisting of a path $x_1, \ldots, x_{t-1}$ and an edge $x_{t-2}x_t$, then the pair (G, C) has a CDC containing the circuit C.*

Proof. Induction on $|V(G)|$. Let $C = v_1v_2, \ldots, v_rv_1$ be the given circuit and T be the small-ended Hamilton path or small-ended spanning Y-tree with an end-vertex x_1 such that $x_1v_1 \in E(G)$. By Lemma 2.2, either G has a circuit C' with $V(C) = V(C')$ and $E(C) \neq E(C')$ or G has a path $P = v_1v_2, \ldots, v_jx_h$ with $V(P) = V(C) \cup \{x_h\}$ for some vertex x_h of T. The path P extends C to a longer circuit $C' = v_1v_2, \ldots, v_jx_hTx_1v_1$.

Let $G' = G - (E(C) - E(C'))^s$. In either case, the reduced pair (G', C') inherits the same property from (G, C): $G' - V(C')$ has either a small-ended Hamilton path or a small-ended spanning Y-tree $T - V(P)$.

By applying induction, let $\mathcal{F}'$ be a CDC of the suppressed graph G' with $C' \in \mathcal{F}'$. Hence, $\mathcal{F} = \mathcal{F}' - C' + \{C'\Delta C,\ C\}$ is a CDC containing the circuit C. ■

3. GIRTH REQUIREMENT FOR COUNTEREXAMPLE TO SCDC

Definition 3.1. *Let g_2 be a largest integer such that, for every pair (G, C) and for some edge e contained in a circuit D of $G - V(C)$ of length less than g_2, the fact that $G - e$ has a CDC containing C implies that G has a CDC containing C.*

Or, simply, if a pair (G, C) is a smallest counterexample to the SCDC conjecture, then the girth of $G - V(C)$ is at least g_2.

Lemma 3.2. *Let G be a cubic graph, C be a given circuit of G and $e \in E(G - V(C))$. Assume that $G - \{e\}$ has a CDC containing the given C but G does not. Then, the edge e is not contained in any circuit of $G - V(C)$ of length ≤ 5. That is, $g_2 \geq 6$.*

Proof. We make a proof by contradiction. Let $D = v_0, \ldots, v_rv_0$ be a circuit of length $r + 1 \leq 5$ contained in $G - V(C)$ and $e = v_0v_r$. In the graph $G - \{e\}$, let $\mathcal{F}$ be a CDC of $G - \{e\}$ containing the circuit C with $|\mathcal{F}|$ as *large* as possible.

A member of $\mathcal{F} - \{C\}$ is denoted by $C_{\alpha,\beta}$ if one component of $C_{\alpha,\beta} \cap D$ is the segment $v_\alpha, \ldots, v_\beta$ $(0 \le \alpha < \beta \le r)$ of $D - \{e\}$. Let $\mathcal{F}' = \{C_{\alpha,\beta} : 0 \le \alpha < \beta \le r\}$ be the set of all such circuits. Note that $|\mathcal{F}'| \le r + 1 \le 5$. And it is evident that

(1) either there is a member $C_{0,r} \in \mathcal{F}'$,
(2) or there are two members $C_{0,\alpha}, C_{\beta,r} \in \mathcal{F}'$ where $0 < \beta \le \alpha < r$.

Case 1: There is a member $C_{0,r} \in \mathcal{F}'$.
In $\mathcal{F}$, replace $C_{0,r}$ with two circuits $\{D_1, D_2\}$ of $C_{0,r} + e$ that cover e twice and all edges of $C_{0,r}$ once. The resulting family of circuits is a CDC for the entire graph G.

Case 2: There are two members $C_{0,\alpha}, C_{\beta,r} \in \mathcal{F}'$ where $0 < \beta \le \alpha < r$.

Claim: v_0, v_r are in the same component of the symmetric difference $E(C_{0,\alpha}) \triangle E(C_{\beta,r})$

We make a proof by contradiction to the claim. Assume that v_0, v_r are in different components of the symmetric difference $E(C_{0,\alpha}) \triangle E(C_{\beta,r})$.

Let H_1 be the subgraph of G induced by edges of $E(C_{0,\alpha}) \cup E(C_{\beta,r})$.

One is able to color all edges of the suppressed cubic graph ${H_1}^s$ with three-colors: red for all edges of $E(C_{0,\alpha}) \cap E(C_{\beta,r})$, and blue and yellow alternatively for the symmetric difference $E(C_{0,\alpha}) \triangle E(C_{\beta,r})$ such that the edges containing v_0, v_r are all colored with blue (because v_0, v_r are in different components of $E(C_{0,\alpha}) \triangle E(C_{\beta,r})$).

Let $D_{red,blue}$ (and $D_{red,yelloe}$) be the even subgraphs of H_1 induced by edges colored with red and blue (red and yellow, respectively).

In $\mathcal{F}$, replace $C_{0,\alpha}, C_{\beta,r}$ with the circuit decompositions of each of $\{D_{red,blue}, D_{red,yellow}\}$.

If the circuit decomposition of $D_{red,blue}$ has more than one circuit, then the resulting family of circuits is a CDC for the graph $G - e$. But it is larger than $\mathcal{F}$. This contradicts that $\mathcal{F}$ is the largest one.

If the circuit decomposition of $D_{red,blue}$ has only one circuit, then it can be dealt with by the same method as Case 1 since it contains both vertices v_0 and v_r. This completes the proof of the claim.

By the claim, v_0, v_r are contained in the same component of the symmetric difference $E(C_{0,\alpha}) \triangle E(C_{\beta,r})$

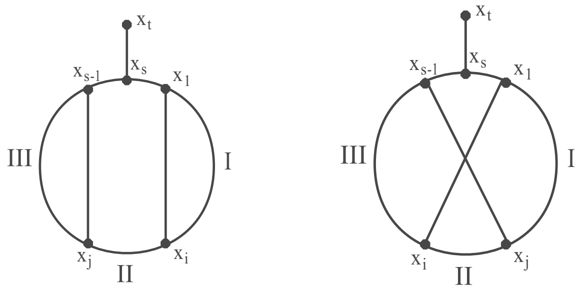
Let H_2 be the subgraph of G induced by edges of $E(C_{0,\alpha}) \cup E(C_{\beta,r}) \cup \{e\}$.

One is able to color all edges of the suppressed cubic graph ${H_2}^s$ with 3-colors: Red for all edges of $E(C_{0,\alpha}) \cap E(C_{\beta,r})$ and the edge e, and blue-yellow alternatively for the symmetric difference $E(C_{0,\alpha}) \triangle E(C_{\beta,r})$.

Similarly, let $D_{red,blue}$ (and $D_{red,yelloe}$) be the even subgraphs of H induced by edges colored with red and blue (red and yellow, respectively).

In $\mathcal{F}$, replace $C_{0,\alpha}, C_{\beta,r}$ with two even subgraphs $\{D_{red,blue}, D_{red,yellow}\}$. The resulting family of circuits is a CDC for the entire graph G. ■

Remark. Although the SCDC and CDC are different problems and the description of g_2 in Definition 3.1 is even more complicated, proofs in some earlier articles, such as [13], [25], and [14], still can be adapted for the girth g_2 requirement for the SCDC conjecture. Note that the adaption of those proofs is relatively long and is therefore not included in this article.



(a) Case 1: $i < j$ (b) Case 2: $i > j$

FIGURE 5. The local structure of H.

4. $G - V(C)$ HAS A HAMILTON PATH OR Y-TREE (THEOREM 1.10)

Note that if $G - V(C)$ contains a Hamilton path P, then either P is small ended or $G - V(C)$ contains a circuit (since each endvertex of P must adjacent to some other vertex in P).

Theorem 4.1. *For a pair (G, C), if $H = G - V(C)$ contains a Hamilton path and is of order less than $3g_2 - 3$, then G has a CDC containing the circuit C.*

Proof. Suppose that (G, C) is counterexample to the theorem with $|V(H)|$ as small as possible.

Claim 1. *We claim that every Hamilton path in H is not small-ended.*

If there exists a small-ended Hamilton path P in H, then the theorem is true by Theorem 2.4 and (G, C) is not a counterexample.

Since a Hamilton path P is acyclic, every circuit of H contains a chord of P. Furthermore, by deleting a chord e of P, the pair $(G - e^s, C)$ remains satisfying the theorem but smaller. Thus, $(G - e, C)$ has a CDC containing C. By the definition of g_2, we have the following conclusion.

Claim 2. *The girth of $H = G - V(C)$ is at least g_2.*

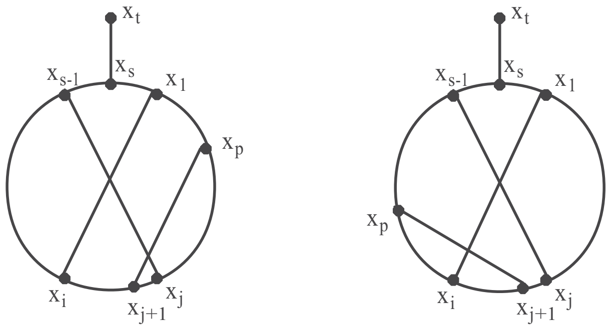
Claim 2 will be used frequently in the remaining part of the proof.

Let $P = x_1x_2, \ldots, x_i, \ldots, x_s, \ldots, x_t$ be any Hamilton path in H with $N(x_1) = \{x_2, x_i, x_s\}$ and $2 < i < s \leq t < 3k - 3$. If $s = t$, then H contains a Hamilton circuit and any vertex on the circuit having a neighbor in C is a small ended of some Hamilton path. Thus, assume that $s < t$. We choose such a Hamilton path that s is maximized. By Claim 1, all neighbors of x_{s-1} are contained in P. Furthermore, by the maximality of the integer s, $N(x_{s-1}) = \{x_j, x_{s-2}, x_s\}$, where $j < s - 2$ (see Fig. 5.)

Notation: Let $p < q$ be two positive integers, denote by $|(p,\ q)|$ the number of integers contained in this open interval. For example, $|(3,\ 5)| = 1$.

Case 1: $i < j$. By the definition of g_2 ($g_2 = k$), we know that $|(1, i)| \geq k - 2$, $|(i,\ j)| \geq k - 5$ and $|(j, s - 1)| \geq k - 2$. Therefore, $s \geq (k - 2) + (k - 5) + (k - 2) + 5 = 3k - 4$ and $t \geq s + 1 \geq 3k - 3$.

Case 2: $i > j$. The fact that $|(j, i)| \geq k - 5 \geq 1$ comes from the circuit $x_j, \ldots, x_ix_1x_sx_{s-1}x_j$. The Hamilton path $x_{j+1}, \ldots, x_{s-1}x_j, \ldots, x_1x_s, \ldots, x_t$ implies that



(a) $1 < p < j$ (b) $j + 2 < p < s - 1$

FIGURE 6. The local structure of H.

$N(x_{j+1}) \subset \{x_1, \ldots, x_s\}$ by the maximality of s, and denote $x_p = N(x_{j+1}) \backslash \{x_j,\ x_{j+2}\}$ (see Fig. 6).

Case 2(a): $x_p \in \{x_2, \ldots, x_{j-1}\}$. The circuit $x_1, \ldots, x_s$ and two chords x_1x_i, $x_{j+1}x_p$ imply that $s \geq 3k - 4$ and $t \geq 3k - 3$.

Case 2(b): $x_p \in \{x_{j+2}, \ldots, x_{s-2}\}$. The circuit $x_1 \ldots x_s$ and two chords x_jx_{s-1}, $x_{j+1}x_p$ imply that $s \geq 3k - 4$ and $t \geq 3k - 3$.

In either case, we see a contradiction that $t \geq 3k - 3$. Immediately, we get the next corollary by Lemma 3.2. ■

Corollary 4.2. *For a pair $(G,\ C)$, if $H = G - V(C)$ contains a Hamilton path and is of order less than 15, then (G, C) has a CDC containing the circuit C.*

Theorem 4.3. *For a pair $(G,\ C)$, if $H = G - V(C)$ contains a spanning Y-tree and is of order less than $3g_2 - 3$, then $(G,\ C)$ has a CDC containing the circuit C.*

Proof. The theorem can be proved by a proof similar to that of Theorem 2.4 if there exists a small-ended spanning Y-tree in H. Thus, we may assume that any spanning Y-tree in H has no small-ended vertex.

Let $Y = x_1x_2 \ldots x_{t-2}x_{t-1} + x_{t-1}x_t$ be any spanning Y-tree with $N(x_1) = \{x_2, x_i, x_s\}$ with $2 < i < s \leq t < 3k - 3$. If $s \in \{t - 1,\ t\}$, then $G - V(C)$ has a Hamilton path and is proved by Theorem 4.1. So assume that $s < t - 2$. We can choose such a Y-tree that s is maximized.

With a similar proof of Theorem 4.1 (detail omitted), we can prove that

$$s \geq 3k - 4,$$

which implies that $t \geq (3k - 4) + 3 = 3k - 1$ and contradicts the fact $t < 3k - 3$. ■

Corollary 4.4. *For a pair $(G,\ C)$, if $H = G - V(C)$ contains a spanning Y-tree and is of order less than 15, then $(G,\ C)$ has a CDC containing the circuit C.*

The combination of Corollaries 4.2 and 4.4 yields Theorem 1.10

The next corollary can be derived directly from the above two theorems, and slightly improves an early result by Fleischner and Häggkvist [12] for $|V(H)| \leq 4$ and H is connected.

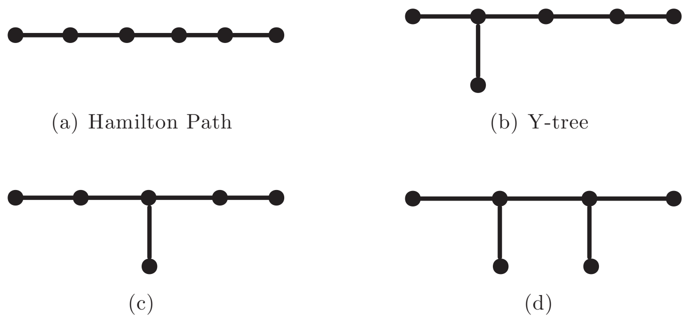


FIGURE 7. All possible spanning trees with six vertices.

Corollary 4.5. *For a pair (G, C), if $H = G - V(C)$ is connected and of order at most 5, then (G, C) has a CDC containing the circuit C.*

Proof. Since $|V(H)| \leq 5$ and G is cubic, every spanning tree of H is either a Hamilton path or a Y-tree. Then, by applying the above two theorems, we may find a CDC containing the circuit C. ■

5. $H = G - V(C)$ IS CONNECTED (THEOREM 1.11)

The following lemma will be used in the proof of Theorem 1.11.

Lemma 5.1 (Fleischner and Häggkvist [12]). *For a pair (G, C) with $|V(G) - V(C)| \leq 2$, and in the case of $|V(G) - V(C)| = 2$, the distance between two vertices of $V(G) - V(C)$ is 3. Then, G has a CDC containing C.*

Now, we are ready to prove Theorem 1.11.

Proof of Theorem 1.11. Induction on $|V(H)| = |V(G) - V(C)|$.
By Corollary 4.5, it is sufficient to consider H of order 6. Let $C = v_1v_2, \ldots, v_rv_1$ be the circuit and $V(H) = \{x_1, \ldots, x_6\}$. ■

Claim 1. *H does not contain a Hamilton circuit.*

Since G is connected, there is an edge x_1v_j joining H and C. If H contains a Hamilton circuit $x_1, \ldots, x_6x_1$, then H has a small-ended Hamilton path $x_1, \ldots, x_6$ and a strong CDC is obtained in Theorem 2.4.
Hence, by Lemma 3.2, we may assume the following.

Claim 2. *H is acyclic (see Fig. 7).*

Since H is acyclic (by Claim 2) and G is cubic, each leaf of H must be adjacent to some vertex of C. Let x_1 be a leaf of H such that $x_1v_1 \in E(G)$ for some vertex $v_1 \in V(C)$. By the Lemma 2.2, either G has a circuit C' with $V(C) = V(C')$ and $E(C) \neq E(C')$ or G has a path $P = v_1v_2, \ldots, v_jx_h$ with $V(P) = V(C) \cup \{x_h\}$ for some vertex $x_h \in V(H)$, which extends C to a longer circuit $C' = v_1v_2, \ldots, v_jx_h, \ldots, x_1v_1$. In either case, the reduced pair (G', C') has one of the following properties, where G' is the suppressed cubic graph $G - (E(C) - E(C'))^s$.

(1) $H' = G' - C'$ remains connected and is of order at most 5,
(2) or $|V(H')| = |V(G') - V(C')| = 2$ and the distance between those two vertices is 3.

By applying induction hypothesis or Lemma 5.1, let $\mathcal{F}'$ be a CDC of the suppressed graph G' with $C' \in \mathcal{F}'$. Hence, $\mathcal{F} = \mathcal{F}' - C' + \{C'\Delta C, \ C\}$ is a CDC containing the circuit C.

6. OPEN PROBLEMS

Theorem 6.1 (Fleischner [10], also see [12]). *Let G be a bridgeless cubic graph of order n and C be a circuit of G of length at least $n-1$. Then, G has a CDC containing the circuit C.*

However, the following problem remains open.

Conjecture 6.2 (Fleischner [11]). *Let G be a bridgeless cubic graph of order n and containing a circuit of length at least $n-1$. Then SCDC conjecture is true for G. (That is, G has a CDC containing a circuit C where C is an arbitrary circuit of G.)*

Note that if C is contained in a circuit of length $n-1$, then, by Theorem 6.1, (G, C) has a SCDC. However, it remains open if C is not contained in any circuit of length $n-1$.

Definition 6.3. *Let G be a cubic graph and F be a spanning even subgraph of G. The oddness of F is the number of odd-components of F. The oddness of G is the minimum oddness of all spanning even subgraphs of G.*

It is trivial that G is 3-edge-colorable if and only if it is of zero oddness. Seymour proved [27] that SCDC conjecture holds for zero-oddness graphs.

Note that a cubic graph with a Hamilton path is of oddness at most 2, a graph described in Theorem 4.1 (containing a spanning subgraph consisting of a circuit and path) is of oddness at most 4. Although the CDC conjecture have been verified for oddness 2 or 4 graphs ([19], [18], [16]), the SCDC conjecture remains open for such small-oddness graphs.

Conjecture 6.4. *Let G be a bridgeless graph of oddness at most 2. Then, the SCDC conjecture is true for (G, C), where C is a circuit of G.*

Conjecture 6.2 is obviously an extreme case of Conjecture 6.4.

For a specified circuit, the following is a weak version of Conjecture 6.4.

Conjecture 6.5. *Let G be a bridgeless graph containing a spanning even subgraph F of oddness at most 2. Then, the SCDC conjecture is true for (G, C), where C is a connected component of F.*

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