Nowhere-Zero 3-Flows in Products of Graphs

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Abstract: A graph G is an odd-circuit tree if every block of G is an odd length circuit. It is proved in this paper that the product of every pair of graphs G and H admits a nowhere-zero 3-flow unless G is an odd-circuit tree and H has a bridge. This theorem is a partial result to the Tutte's 3-flow conjecture and generalizes a result by Imrich and Skrekovski [7] that the product of two bipartite graphs admits a nowhere-zero 3-flow. A byproduct of this theorem is that every bridgeless Cayley graph $G = Cay(\Gamma, S)$ on an abelian group Γ with a minimal generating set S admits a nowhere-zero 3-flow except for odd prisms. J Graph Theory 50: 79–89, 2005

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1. INTRODUCTION

The work in this paper is motivated by the following two conjectures.

Conjecture 1.1 (Tutte [12], [2]). *Every bridgeless graph without* 3*-edge-cut admits a nowhere-zero* 3*-flow.*

Note that Conjecture 1.1 is equivalent to the statement that *every* 5*-edge-connected* [9], 5*-regular* [10, 17] *graph admits a nowhere-zero* 3*-flow.*

Conjecture 1.2. *Let G be a* 4*-edge-connected graph such that every edge of G is contained in a circuit of length at most* 4*. Then G admits a nowhere-zero* 3*-flow.*

Conjecture 1.2 is true for the 4-flow problem by a theorem of Catlin [3] without the requirement of 4-edge-connectivity, and by a theorem of Jaeger [8] without the requirement of small circuits, but remains open for 3-flow.

The following theorem is a partial result related to Conjecture 1.2.

Theorem 1.3 (Imrich and Skrekovski [7]). *Let G and H be two graphs. Then $G \times H$ admits a nowhere-zero* 3*-flow if both G and H are bipartite.*

Theorem 1.3 is to be generalized by the main result (Theorem 1.5) of the paper.

Definition 1.4. *A connected graph G is a circuit-tree if every block of G is a circuit. A circuit-tree G is odd if every block of G is of odd length.*

The set of all odd-degree vertices of a graph G is denoted by $O(G)$.

Theorem 1.5. *Let G and H be two connected non-trivial graphs with $|O(G)| \leq |O(H)|$. If the graphs G and H do not have the structure that G is an odd-circuit-tree and H has a bridge, then their product $G \times H$ admits a nowhere-zero* 3*-flow.*

It can be proved (Lemma 3.6) that, for a 2-edge-connected graph G, G contains no circuit of even length if and only if it is an odd-circuit-tree. Immediate corollaries of Theorem 1.5 are that *the product of any pair of bridgeless graphs admits a nowhere-zero* 3*-flow,* and, *the product of any pair of graphs both containing circuits of even lengths admits a nowhere-zero* 3*-flow.* One may also notice that the converse to Theorem 1.5 does not hold since the products of some odd-circuit-trees and some graphs with bridges do admit nowhere-zero 3-flows.

The following result is another corollary of Theorem 1.5.

Corollary 1.6. *Let Γ be an abelian group and S be a minimal generating set of the group Γ. If the Cayley graph $G = Cay(\Gamma, S)$ is of degree at least* 2*, then G admits a nowhere-zero* 3*-flow except for only one case: $S = \{\alpha, \beta\}$ where $|\alpha| = 2$, $|\beta|$ is odd.*

Corollary 1.6 is related to a conjecture [1] that *every Cayley graph admits a nowhere-zero* 4*-flow* (the best result up-to-date to this conjecture is for solvable groups, see [1]). The proof of Corollary 1.6 is straightforward. The degree of G must be odd for otherwise a nowhere-zero 2-flow can be found in G (by Lemma 3.1). Hence, S must contain an element α with $|\alpha| = 2$. Since S is a minimal generating set of the abelian group Γ, the graph G is the product of two smaller Cayley graphs $Cay(\langle S - \alpha \rangle, (S - \alpha))$ and a single edge K_2. Corollary 1.6 follows directly by applying Theorem 1.5.

2. NOTATION AND TERMINOLOGY

A *circuit* is a connected 2-regular subgraph. A circuit of even length (or, odd length, respectively) is called an *even circuit* (or an *odd circuit*, respectively). An *even subgraph* is a union of edge-disjoint circuits. An even subgraph is *eulerian* if it is connected. An edge e that is not contained in any circuit of G is called a *bridge* of G.

The set of all *odd degree vertices* of a graph G is denoted by $O(G)$. For a vertex v of G, the set of edges incident with v is denoted by $E(v)$. The set of vertices of G with degree precisely μ is denoted by V_μ.

Let D be an orientation of a graph G. For a vertex v, $E^+(v)$ is the set of all directed edges with tails at v, and $E^-(v)$ is the set of all directed edges with heads at v.

Let G be a graph. The *underlying graph* of G, denoted by $\overline{G}$, is the graph obtained from G by replacing every maximal induced path a single edge. That is, $\overline{G}$ is a graph without degree 2 vertex to which G is homeomorphic.

An integer flow of a graph G is an ordered pair (D,f) where D is an orientation of $E(G)$ and $f : E(G) \mapsto Z$ and Z is the set of all integers such that

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$$

for every vertex v of G.

An integer flow (D,f) is a k-flow if $|f(e)| < k$ for every edge of G. The *support* of a k-flow (D,f) of G is the following set of edges

$$\text{supp}(f) = \{e \in E(G) : f(e) \neq 0\}.$$

A k-flow (D,f) of G is *nowhere-zero* if $\text{supp}(f) = E(G)$.

Let A be an additive (abelian) group. An *A-flow* (D,f) of a graph G consists of an orientation D and a mapping $f : E(G) \mapsto A$ such that

$$\sum_{e \in E^+(v)} f(e) \equiv \sum_{e \in E^-(v)} f(e)$$

for every vertex $v \in V(G)$. The support of a group A-flow and a nowhere-zero A-flow is defined similarly as integer flows.

Readers may immediately notice that if a graph G admits a nowhere-zero k-flow (D,f) then G admits a nowhere-zero k-flow (D',f') for any orientation D' of G since the changes of signs of f for all edges with opposite orientations under D and D' yield a nowhere-zero k-flow (D',f'). It is the same for group A-flows.

Note that a graph G admits a nowhere-zero integer-valued k-flow if and only if G admits a nowhere-zero Z_k-flow (by a theorem of Tutte [14] and [15] or see [18]: Theorem 1.3.3 and Theorem 2.2.3). We sometime work with group Z_3-flows instead of integer 3-flows for the sake of technical convenience.

An orientation D of a graph G is called a mod-3-*orientation* if

$$|E^+(v)| - |E^-(v)| \equiv 0 \bmod (3).$$

Let G and H be two graphs. The product of G and H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$, and two vertices (g,h), (g',h') of $G \times H$ are adjacent to each other if either g and $g' \in V(G)$ are adjacent in G or h and $h' \in V(H)$ are adjacent in H.

Let $g_0 \in V(G)$ and $h_0 \in V(H)$. In the product $G \times H$, the subgraph of $G \times H$ induced by all vertices $\{(g_0,h) : h \in V(H)\}$ is called an H-layer, and the subgraph of $G \times H$ induced by all vertices $\{(g,h_0) : g \in V(G)\}$ is called a G-layer.

3. LEMMAS

A. Basic Lemmas for Flows

Lemma 3.1 (Tutte [16]). *A graph G admits a nowhere-zero 2-flow if and only if the graph G is even (that is, each component of G is eulerian).*

Lemma 3.2 ([13] or see [18] Lemma 4.1.2). *A graph G admits a nowhere-zero 3-flow if and only if G admits a* mod-3-*orientation.*

Lemma 3.3 (Tutte [14]). *A cubic graph G admits a nowhere-zero 3-flow if and only if G is bipartite.*

Lemma 3.4. *Let G be a graph. If the subgraph of G induced by V_3 is not bipartite, then G does not admit a nowhere-zero 3-flow.*

Proof. If the subgraph of G induced by V_3 is not bipartite, then G cannot have a mod-3-orientation. (Lemma 3.2 is applied here.) ■

Lemma 3.5 (Seymour [11]). *Let G be a bridgeless graph. Then G admits a 3-flow (D,f) and contains an even subgraph C such that* $\operatorname{supp}(f) \cup E(C) = E(G)$.

B. CIRCUIT TREES AND EVEN CIRCUITS

Lemma 3.6. *A graph G contains no even circuit if and only if every block of G is an odd circuit or a single edge.*

The proof is quite simple and is left as an exercise for readers.

C. VERTEX SPLITTING

Definition 3.7 (Vertex Splitting)**.** *Let v be a vertex of G and $e_1, e_2 \in E(v)$ having ends v_1 and v_2, respectively, different from v. The graph $G_{[v;e_1,e_2]}$ is obtained from G by deleting e_1 and e_2 and adding a new edge v_1v_2.*

Lemma 3.8 (Fleischner [4], or see [5] [6] [11] [18])**.** *Let G be a 2-edge-connected graph and $v \in V(G)$ with $d(v) \geq 4$. Then there are edges $e_1, e_2 \in E(v)$ such that $G_{[v;e_1,e_2]}$ remains 2-edge-connected.*

Lemma 3.9. *Let G be a connected graph and $O(G)$ be the set of all odd vertices of G.*

(1) *Suppose that $O(G) \neq \emptyset$. Then, after a series of vertex splitting operations, we can obtain a new graph from G that is a disjoint union of $|O(G)|/2$ paths of length at least one.*
(2) *Suppose that $O(G) = \emptyset$. Then, after a series of vertex splitting operations, we can obtain a new graph from G that is a circuit of length $|V(G)|$.*

Proof. For (1), construct a new graph G^* from G by adding a new vertex x and new edges xv_i for every $v_i \in O(G)$. Let T be an Euler tour of G^*. Deleting x from T, we obtain a set of $(|O(G)|/2)$ edge-disjoint trails of length at least one. Applying vertex splitting operations along those trails until the degree of every vertex is 1 or 2.

(2) is similar to (1) by splitting high degree vertices along an Euler tour T. ■

Lemma 3.10. *If an eulerian G is not an odd-circuit tree, then, after a series of splitting operations, we can obtain either an even circuit (if $|V(G)| =$ even) or a circuit tree consisting of precisely two even circuits (if $|V(G)| =$ odd).*

Proof. **Case 1.** $|V(G)| =$ even, then G can be split to a circuit of length $|V(G)|$ by Lemma 3.9 (2).

Case 2. $|V(G)| =$ odd. Let G be a counterexample with $|E(G)|$ as small as possible. By Lemma 3.6, G contains an even circuit C.

Subcase 2.1. We claim that every vertex of $V(C)$ is either of degree 2 or is a cut-vertex of G. Assume that $v \in V(C)$ is of degree greater than 2 and is not a cut-vertex of G. Let $e_1, e_2 \in E(v) - E(C)$. Then, after splitting e_1, e_2 away from v, the resulting graph $G' = G_{[v;e_1,e_2]}$ remains connected and has fewer number of edges than G. So, the smaller graph G', which contains an even circuit C, can be further split to a circuit tree consisting of precisely two even circuits.

Subcase 2.2. We claim that the even circuit C cannot intersect with any odd circuit. Assume that C' is an odd circuit that $V(C) \cap V(C') \neq \emptyset$. By Subcase 2.1, the intersection of these circuits must be a cut vertex v of G. Let $e_3 \in E(C) \cap E(v)$ and $e_4 \in E(C') \cap E(v)$. Then, after splitting e_3, e_4 away from v, the resulting graph $G'' = G_{[v;e_3,e_4]}$ remains connected and has fewer number of edges than G, and the new circuit resulted by the combination of two circuits C and C' is of even length. So, the smaller graph G'', which contains an even circuit, can be further split to a circuit tree consisting of precisely two even circuits.

Subcase 2.3. By Subcase 2.2, every circuit intersecting with C must be of even length. Note that the even circuit C was chosen arbitrarily. Therefore, by Subcase 2.2 again to every even circuit of G, every circuit of G is of even length since G is connected. By Subcase 2.1, the graph is a circuit tree with every block as an even circuit.

Subcase 2.4. We claim that any even circuit C cannot intersect with two other circuits. Assume that C intersects with two circuits C_1 and C_2 (both are of even lengths, by Subcase 2.2). By Subcase 2.1, the intersection of the circuits C and C_1 must be a cut vertex v of G. Let $e_5 \in E(C) \cap E(v)$ and $e_6 \in E(C_1) \cap E(v)$. Then, after splitting e_5, e_6 away from v, the resulting graph $G''' = G_{[v;e_5,e_6]}$ remains connected and has fewer number of edges than G. So, the smaller graph G''', which contains a circuit C_2 of even length, can be further split to a circuit tree consisting of precisely two even circuits.

By Subcases 2.3 and 2.4, it is obvious that the circuit tree G has only two blocks, each of which is a circuit of even length. ■

Lemma 3.11. *Let G and H be two graphs and $v \in V(G)$ with $d(v) \geq 3$ and $e_1, e_2 \in E(v)$. Then $G \times H$ admits a nowhere-zero 3-flow if $G_{[v;e_1,e_2]} \times H$ admits a nowhere-zero 3-flow.*

Proof. Obvious. ■

4. PROOF OF THE MAIN THEOREM

Consider a counterexample $G \times H$ to the theorem with the fewest number of edges. The proof is to be divided into several parts.

In the first part, we are to find nowhere-zero 3-flows for the product of certain graphs (circuits, paths, and circuit trees) in several lemmas. Though those graphs are too special, lemmas proved in this part will be useful in Part Two.

In the second part of the proof, we are to prove that one factor, say G, must be a circuit of odd length. The proof is outlined as follows. By applying the vertex splitting method (defined in Subsection 3C) and some lemmas in Part One, we first show that one of $\{G, H\}$ is an even subgraph while another one is not. Furthermore, we will reduce the factors G and H to be some of those special graphs considered in Part One. The existence of nowhere-zero 3-flows follows by applying those lemmas proved in Part One.

In the third (the last) part of the proof, the product $G \times H$ is to be decomposed into a few subgraphs. The proof is to be completed here by showing that each of these subgraphs admits a nowhere-zero 3-flow. In this part, the 6-flow theorem of Seymour [11] is to be applied to the non-eulerian factor H.

A. Part One: Products of Special Graphs

Lemma 4.1. *If G and H are paths of length ≥ 1, then $G \times H$ admits a nowhere-zero 3-flow.*

Proof. Since $G \times H$ is planar, we can apply a theorem of Tutte [15] that a planar graph is face k-colorable if and only if G admits a nowhere-zero k-flow. The faces of $G \times H$ can be easily colored as follows: red and blue alternatively for square faces, and yellow for the exterior face. ■

Lemma 4.2. *Suppose that G is a circuit and H is a path. Then $G \times H$ admits a nowhere-zero Z_3-flow if and only if G is a circuit of even length. Furthermore, if G is a circuit of even length and H is a path, then for any given vertex $v \in V(G)$, the product $G \times H$ admits a nowhere-zero 3-flow (D,f) such that the H-layer $\{v\} \times H$ is oriented under D as a directed path with $f(e) = 1$ for every edge e of $\{v\} \times H$.*

Proof. Let $g_0 \cdots g_{2k-1} g_0$ be the even circuit G and $h_0 \cdots h_t$ be the path H. Assign a mod-3-orientation D to $G \times H$ as follows.

(1) In the G-layer $G \times \{h_0\}$, the vertex $(g_{2\mu}, h_0)$ *dominates* both $(g_{2\mu-1}, h_0)$ and $(g_{2\mu+1}, h_0)$ for every $\mu = 0, \ldots, k-1$ (mod $(2k)$);
(2) In the G-layer $G \times \{h_t\}$, the vertex $(g_{2\mu}, h_0)$ *is dominated* by both $(g_{2\mu-1}, h_0)$ and $(g_{2\mu+1}, h_0)$ for every $\mu = 0, \ldots, k-1$ (mod $(2k)$);
(3) Every other G-layer $G \times \{h_r\}$ ($r \neq 0$ or t) is oriented as a directed circuit (in either direction);
(4) For every $\mu = 0, \ldots, k-1$, the H-layer $\{g_{2\mu}\} \times H$ is oriented as a directed path from $(g_{2\mu}, h_0)$ to $(g_{2\mu}, h_t)$;
(5) For every $\mu = 0, \ldots, k-1$, the H-layer $\{g_{2\mu+1}\} \times H$ is oriented as a directed path from $(g_{2\mu+1}, h_t)$ to $(g_{2\mu+1}, h_0)$.

It is easy to see that D is a mod-3-orientation. Let $f : E(G \times H) \mapsto \{1\}$. Obviously (D,f) is a Z_3-flow of the graph.

If G is a circuit of odd length, then the product contains an odd length circuit that consists of degree 3 vertices, which cannot admit nowhere-zero 3-flow (by Lemma 3.4). ■

Lemma 4.3. *Suppose that G is a circuit tree consisting of two even circuits and H is a path. Then $G \times H$ admits a nowhere-zero* 3*-flow.*

Proof. Let $C_1 = u_0u_1 \cdots u_au_0$ and $C_2 = v_0v_1 \cdots v_bv_0$ be circuits of G with the cut-vertex $x = u_0 = v_0$. By Lemma 4.2, $C_i \times H$ admits a nowhere-zero Z_3-flow (D,f_i) for each i such that both $\{u_0\} \times H$ and $\{v_0\} \times H$ are directed paths under the orientation D with the same flow-value 1. Hence, (D,f_1+f_2) is a nowhere-zero Z_3-flow of $G \times H$. ■

B. Part Two

Lemma 4.4. *Let $G \times H$ be a counterexample to Theorem* 1.5 *with the fewest number of edges. If $|O(G)| \leq |O(H)|$, then G must be a circuit of odd length and H must be cubic.*

Proof. I. We claim that *either $O(G)$ or $O(H) = \emptyset$, but not both.* If neither $O(G)$ nor $O(H) = \emptyset$, then by Lemma 3.9 (1), the graphs G and H can be split to graphs G^* and H^*, each of which is a disjoint union of paths. By Lemma 4.1, each component of $G^* \times H^*$ admits a nowhere-zero 3-flow, so is $G^* \times H^*$. By Lemma 3.11, $G \times H$ admits a nowhere-zero 3-flow as well. This contradicts that $G \times H$ is a counterexample. So assume that $O(G) = \emptyset$. But $O(H)$ cannot be empty, for otherwise, both G and H are eulerian, so is $G \times H$, which admits a nowhere-zero 2-flow (by Lemma 3.1).

II. We claim that *G is an odd circuit tree*. Assume that G is eulerian but not an odd-circuit tree. By Lemma 3.10, the graph G can be split to a graph G^* that is either a circuit of even length or a circuit tree consisting of precisely two circuits of even lengths. By Lemma 3.9 (1), let H^* be the graph obtained from H by vertex-splitting operations such that H^* is a disjoint union of paths. By Lemma 4.2 or Lemma 4.3, the graph $G^* \times H^*$ admits a nowhere-zero 3-flow, so is $G \times H$ (by Lemma 3.11), a contradiction.

III. We claim that *G is an odd circuit.* Since $G \times H$ is a smallest counterexample to the theorem and G is an odd-circuit tree, the another factor H must be bridgeless. If G is not a circuit, by Lemma 3.9 (2), G can be split further to an odd circuit G^* with $|E(G^*)| < |E(G)|$. Here, $G^* \times H$ is smaller than $G \times H$ and, therefore, admits a nowhere-zero 3-flow, so is $G \times H$ (by Lemma 3.11). Thus, we have that G is an odd circuit and H is bridgeless.

IV. We claim that $\delta(H) > 2$. Assume that $d(h) = 2$ for some vertex $h \in V(H)$ with $e_1, e_2 \in E(h)$. Note that h becomes an isolated vertex of $H_{[h;e_1,e_2]}$. Then $G \times H_{[h;e_1,e_2]}$ is the union of two disconnected parts: one is a circuit $G \times \{h\}$, while the another one $G \times [H_{[h;e_1,e_2]} - \{h\}]$ is smaller than the smallest counterexample $G \times H$. Therefore, both parts admit nowhere-zero 3-flows, so is their union $G \times H$.

V. By Lemma 3.8, vertices of H with degree ≥ 4 can be split. Hence, we may assume that H is cubic, for otherwise, by Lemma 3.11, $G \times H$ is not a smallest counterexample. ■

C. Part Three—The Final Step

The proof of the theorem is to be completed in this part.

Let $G \times H$ be a counterexample to the theorem with the fewest number of edges. By Lemma 4.4, the factor G is a circuit of odd length and the factor H is a bridgeless, cubic graph. Let the vertex set of the bridgeless cubic graph H be $\{h_1, \ldots, h_n\}$ and let the circuit G be $g_0 g_1 \cdots g_m g_0$ (m is even).

In the product $G \times H$, for the sake of convenience, a G-layer $G \times \{h_i\}$ is called the i-th G-layer; and an H-layer $\{g_j\} \times H$ is called the j-th H-layer.

C(1). Strategy. Let (D_{H_j}, f_{H_j}) be a 3-flow of the j-th H-layer $H \times \{g_j\}$, and let $B_j = E(H \times \{g_j\}) - \mathrm{supp}(f_{H_j})$—the subset of edges excluded from the support of the 3-flow (D_{H_j}, f_{H_j}) in $H \times \{g_j\}$.

We will construct a subgraph K of $G \times H$ that consists of all edges of B_j, for every $j \in \{0, \ldots, m\}$ and all G-layers.

The key point in this final step is how to choose those 3-flows (D_{H_j}, f_{H_j}) in H-layers so that the subgraph K admits a nowhere-zero 3-flow (D_K, f_K). If we can do so, then the union of the supports of all 3-flows (D_{H_j}, f_{H_j}) and (D_K, f_K) covers the entire graph $G \times H$.

C(2). 3-flows in *H*-layers. Let (D_H, f_H) be a Z_3-flow of H and C be an even subgraph of H such that (by Lemma 3.5)

(1) $\mathrm{supp}(f_H) \cup E(C) = E(H)$;
(2) The even subgraph C is oriented under D_H as a union of edge-disjoint directed circuits.

Let $B_\alpha = \{e \in E(C) : f_H(e) \equiv \alpha\}$ for each $\alpha \in Z_3$. Let (D_H, f'_H) be a 2-flow of H (by Lemma 3.1) such that $f'_H(e) = 1$ if $e \in E(C)$, and $= 0$ otherwise. Since H is isomorphic to each H-layer, without causing any confusion, (D_H, f_H) and (D_H, f'_H) are also considered as flows in each H-layer. For each $j \in \{0, \ldots, m\}$, the 3-flow (D_{H_j}, f_{H_j}) in the j-th H-layer $\{g_j\} \times H$ is constructed as follows.

$$(D_{H_0}, f_{H_0}) = (D_H, f_H),$$
$$(D_{H_1}, f_{H_1}) = (D_H, f_H - f'_H)$$

and

$$(D_{H_j}, f_{H_j}) = (D_H, f_H + f'_H)$$

for $j = 2, \ldots, m$.

Note that (D_H, f'_H) is a non-negative 2-flow of H with the support $E(C) = B_0 \cup B_1 \cup B_2$. Since the sum is taken in the cyclic group Z_3,

$$\mathrm{supp}(f_{H_0}) = \{g_0\} \times H_0 - \{g_0\} \times B_0,$$
$$\mathrm{supp}(f_{H_1}) = \{g_1\} \times H_1 - \{g_1\} \times B_1,$$

and

$$\mathrm{supp}(f_{H_j}) = \{g_j\} \times H_2 - \{g_1\} \times B_2$$

for every $j = 2, \ldots, m$, where $\{g_i\} \times B_\alpha$ is the copy of B_α in the i-th H-layer $\{g_i\} \times H$.

C(3). The subgraph *K*. Let

$$K = G \times H - \bigcup_{i=0}^{m} \mathrm{supp}(f_{H_i}).$$

By the discussion we had in the Subsubsection 4C(1), it is sufficient to find a nowhere-zero 3-flow in the subgraph K.

Since H is cubic and $B_0 \cup B_1 \cup B_2 = E(C)$, it is obvious that each B_α is a matching and each vertex of C is incident with precisely two edges of $B_0 \cup B_1 \cup B_2$. Thus, every component of K is either a circuit or is homeomorphic with a cubic graph. It is obvious that a circuit component of K must be a G-layer $G \times \{h\}$ that $h \notin V(C)$, which admits a nowhere-zero 2-flow (by Lemma 3.1). Therefore, by Lemma 3.3, it is sufficient to show that every non-circuit component of K is homeomorphic with a cubic bipartite graph. That is, we shall find a proper vertex-2-coloring of the graph $\overline{K'}$, where K' is obtained from K by deleting all circuit-components.

Recall that each D_{H_i} is an orientation of the cubic graph $\{g_i\} \times H$ such that every component of the even subgraph C is oriented as a directed circuit. That is, every edge of $K - \bigcup_{i=1}^{n} E(G \times \{h_i\})$ is oriented as a directed edge under the orientation of the corresponding H-layer.

For each $i = 0, 1$ and $i = 2\mu$ for each $\mu = 1, \ldots, m/2$, the heads and the tails of all directed edges of $\overline{K'}$ originally contained in the i-th H-layer are colored with *red* and *blue*, respectively.

If $m \geq 4$, then for each $i = 2\mu + 1,\ \mu \in \{1, \ldots, m/2 - 1\}$, the heads and the tails of all directed edges of $\overline{K'}$ originally contained in the i-th H-layer are colored with *blue* and *red*, respectively.

Now, we are ready to verify that $\overline{K'}$ is bipartite by showing that the above vertex-2-coloring of $\overline{K'}$ is proper.

If a vertex h_i of H is the tail of a directed edge of B_0 and the head of a directed edge of B_1, then the i-th G-layer $G \times \{h_i\}$ is a circuit of length 2 in $\overline{K'}$ with precisely one red vertex and one blue vertex.

If a vertex h_i of H is the tail of a directed edge of B_0 and the head of a directed edge of B_2, then the i-th G-layer $G \times \{h_i\}$ is a circuit of length m in $\overline{K'}$ consisting of $m/2$ *red* vertices and $m/2$ *blue* vertices, alternatively.

It is similar for all other cases. This completes the proof of the theorem. ∎

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