Edge-Face Chromatic Number and Edge Chromatic Number of Simple Plane Graphs

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Abstract: Given a simple plane graph G, an edge-face k-coloring of G is a function $\phi : E(G) \cup F(G) \rightarrow \{1, \ldots, k\}$ such that, for any two adjacent or incident elements $a, b \in E(G) \cup F(G)$, $\phi(a) \neq \phi(b)$. Let $\chi_e(G), \chi_{ef}(G)$, and $\Delta(G)$ denote the edge chromatic number, the edge-face chromatic number, and the maximum degree of G, respectively. In this paper, we prove that $\chi_{ef}(G) = \chi_e(G) = \Delta(G)$ for any 2-connected simple plane graph G with $\Delta(G) \geq 24$. © 2005 Wiley Periodicals, Inc. J Graph Theory 49: 234–256, 2005

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1. INTRODUCTION

All graphs considered in this paper are finite. For a plane graph G, let $\chi_{\mathrm{e}}(G)$, $\Delta(G)$, and $\delta(G)$ denote *the edge chromatic number, the maximum degree,* and *the minimum degree* of the graph G, respectively. Let $E(G)$, $V(G)$, and $F(G)$ denote *the edge set, the vertex set, and the face set* of G, respectively. An *element* of G is a member of $E(G) \cup F(G)$. Any two elements are *adjacent* if they are either adjacent to or incident with each other in the traditional sense. An *edge-face k-coloring* of a plane graph G is a function $\phi : E(G) \cup F(G) \to \{1, \ldots, k\}$ such that for any two adjacent elements $a,\ b \in E(G) \cup F(G)$, $\phi(a) \neq \phi(b)$.

Let $\chi_{\mathrm{ef}}(G)$ denote the *edge-face chromatic number* of G, i.e., the smallest integer k such that G has an edge-face k-coloring. This concept appears to have first been considered by Jucovič [3] and Fiamčík [2].

In 1975, Melnikov [5] made the following conjecture:

Conjecture 1.1 (Melnikov [5], 1975)**.** *For any simple plane graph G*, $\chi_{\mathrm{ef}}(G) \leq \Delta(G) + 3$.

Waller [10] and Sanders and Zhao [7] proved this conjecture independently. In [1], Borodin proved that for $\Delta(G) \geq 10$, $\chi_{\mathrm{ef}}(G) \leq \Delta(G) + 1$. Also in [1], Borodin proposed the following problem: Characterize those simple plane graphs G having $\chi_{\mathrm{e}}(G) = \chi_{\mathrm{ef}}(G) = \Delta(G)$.

In this paper, we investigate the relationship between $\chi_{\mathrm{e}}(G)$ and $\chi_{\mathrm{ef}}(G)$ for 2-connected simple plane graphs G.

Vizing [9] showed that an improvement to his edge coloring theorem is possible for planar graphs with large maximum degree.

Theorem 1.2 (Vizing [9])**.** *Let G be a simple planar graph. If* $\Delta(G) \geq 8$*, then* $\chi_e(G) = \Delta(G)$.

Remark. Vizing [9] conjectured that every planar graph with maximum degree 6 or 7 is also class one. The case $\Delta = 7$ has been verified by Zhang [11] and Sanders and Zhao [8] independently. The case $\Delta = 6$ remains open.

The main result of this paper is the following:

Theorem 1.3. *For any 2-connected simple plane graph G with* $\Delta(G) \geq 24$, $\chi_{\mathrm{ef}}(G) = \Delta(G)$.

For technical reason, we will prove the following theorem.

Theorem 1.4. *Let* $k \geq 24$ *be an integer and G be a 2-connected simple plane graph. If* $\chi_{\mathrm{e}}(G) \leq k$, *then* $\chi_{\mathrm{ef}}(G) \leq k$.

Theorem 1.3 is a corollary of Theorems 1.2 and 1.4.

Remark. In Theorem 1.3, the hypothesis "2-connected" cannot be weakened to "2-edge-connected." For example, let G be a graph obtained from t triangles $x_iy_iz_ix_i$ $(i = 1, \ldots, t)$ by identifying all vertices x_i as a single vertex. Then G is

2-edge-connected but not 2-connected. It is obvious that $\chi_{\mathrm{ef}}(G) = 2t + 1 = \Delta(G) + 1$.

The proof of Theorem 1.4 follows the standard method, of first identifying a number of reducible configurations (that is, configurations that can not be present in a smallest counterexample), and then using the discharging method to prove that a smallest counterexample cannot exits. However, it is easier to use the discharging method when there are no vertices of degree 2. Thus, we first carry out what we call series parallel operations in order to remove vertices of degree 2, at the expense of possibly introducing parallel edges (although not faces of length 2). After a few preliminaries in Sections 2, we discuss reducible configurations in Section 3 and the structure of the resulting graph after applying the series parallel operations on a smallest counterexample in Section 4, and we complete the proof by using discharging method in Section 5.

2. PRELIMILARIES

A path $v_0v_1 \cdots v_r$ is called a *subdivided edge of length r* if $d(v_i) = 2$ for each $i \neq 0, r$ and both $d(v_0) > 2$ and $d(v_r) > 2$.

For $f \in F(G)$, the *boundary* of f, denoted by $B(f)$, is the set of all edges incident with the face f. Let $C = \{1, 2, \ldots, k\}$ denote the color set. Let $\phi : E(H) \cup F(H) \to C$ be an edge-face k-coloring of a plane graph H. For each vertex $v \in V(H)$, $\phi(v)$ is the set of all colors used by the edges incident with v. Let v, f be a vertex and a face in H, respectively. $d(v)$ and $d(f)$ are the degree of v and the length of f, respectively.

Let $A \subseteq E(H) \cup F(H)$. A *partial edge-face k-coloring* of H on A is a function $\phi : A \to C$ such that every pair of adjacent elements in A receive different colors. For a partial edge-face k-coloring ϕ of H on A, let $\phi(u)$ be the set of colors used by the edges in $A \cap E(H)$, which are incident with the vertex $u \in V(H)$.

If there is no confusion, a face is usually denoted by the sequence of vertices that form the circuit (or cycle) around the face.

The following two lemmas will be used in the proof of Theorem 1.4. The first one is obvious and thus the proof is omitted.

Lemma 2.1. *Let A and B be two finite sets with the same cardinality $n \geq 2$. Then there exists a one-to-one mapping f from A to B such that $f(a) \neq a$ for any $a \in A$.*

For an edge $e = xy$, denote by $r_k(xy)$, the number of edges $x'y'$ adjacent to e such that $d(x') + d(y') \leq k - 1$. Let $F_s = \{f \in F(G) | d(f) \leq (k-1)/2\}$ and let E_s be the set of edges uv in G such that either $d(u) + d(v) \leq r_k(uv) + k - 1$, or else $d(u) + d(v) = r_k(uv) + k$ and uv lies in the boundary of some face in F_s.

Lemma 2.2. *Let G be a simple plane graph and k be a positive integer. Let $S \subseteq E_s \cup F_s$. If there is a partial edge-face k-coloring $\phi : [E(G) \cup F(G)] \backslash S \to C$, then G has an edge-face k-coloring.*

Proof. We first uncolor all edges $e = xy$ with $d(x) + d(y) \leq k - 1$ and all the faces in F_s. Let $e = uv \in E_s \cap S$ with $d(u) + d(v) \leq r_k(uv) + k$. If $d(u) + d(v) = r_k(uv) + k$, then e is adjacent to a face in F_s. Therefore, e is adjacent to at most one colored face and at most $k - 2$ colored edges and hence, there is at least one color available for the edge e. We first color all such edges, and then color each edge $e = uv \in S$ with $k \leq d(u) + d(v) \leq r_k(uv) + k - 1$, since e is adjacent to at most two colored faces and at most $k - 3$ colored edges. Finally, we can further color each edge $e = uv$ with $d(u) + d(v) \leq k - 1$ for the same reason, and color each face with length at most $(k - 1)/2$ since there are at least $(k - 2)(k - 1)/2 = 1$ color available for it. ■

3. REDUCIBLE CONFIGURATIONS

The proof of Theorem 1.4 starts from here. From now on, we use G to denote a smallest counterexample to Theorem 1.4. In this section, we discover some reducible configurations of G. These configurations will be used in the rest of the proof of Theorem 1.4.

Proposition 1 (Configuration A). *(i) Every subdivided edge of G is of length at most* 2.
(ii) Let uvw be a subdivided edge of G of length 2. If uw is not an edge of G, then

$$d(u) \geq k - 2 \quad \textit{and} \quad d(w) \geq k - 2.$$

(*The vertices u and w are called the terminal vertices of the configuration.*)

Proof. (i) Assume that $P = x_1x_2x_3 \cdots x_d$ is a subdivided edge of length $d - 1 \geq 3$ in G. Consider the graph G_1 obtained from G by replacing the path $x_1x_2x_3$ with one edge x_1x_3. Then, G_1 remains 2-connected and simple, and by Theorem 1.2, $\chi_e(G_1) \leq k$. Since G is a smallest counterexample, G_1 has an edge-face k-coloring ϕ with the color set C. Color the edge x_1x_2 with the color $\phi(x_1x_3)$. Then, ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)] \backslash \{x_2x_3\}$. Since $d_G(x_2) = d_G(x_3) = 2$, there are at least $k - (d_G(x_2) + d_G(x_3) - 2 + 2) = k - 4 \geq 1$ colors of C available for the edge x_2x_3. Thus, G has an edge-face k-coloring, a contradiction.

(ii) We prove that $d(u) \geq k - 2$; the proof that $d(w) \geq k - 2$ is similar. Suppose that $d(u) \leq k - 3$. Since $d(v) = 2$, $uv \in E_s$ (defined before Lemma 2.2). Since $uw \notin E(G)$, we may replace the path uvw with an edge uw. The resulting graph, denoted G_1', remains 2-connected and simple and by Theorem 1.2, $\chi_e(G_1') \leq k$. Hence, G_1' has an edge-face k-coloring ϕ. Then, ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)] \backslash \{uv, vw\}$. Since $uv \in E_s$, ϕ can be extended to an edge-face k-coloring of G by Lemma 2.2. ■

Proposition 2. *G does not contain either of configurations illustrated in Figure* 1, *where* $d(v_i) = 2$ *for each i, and* f_1 *and* f_2 *are faces.*

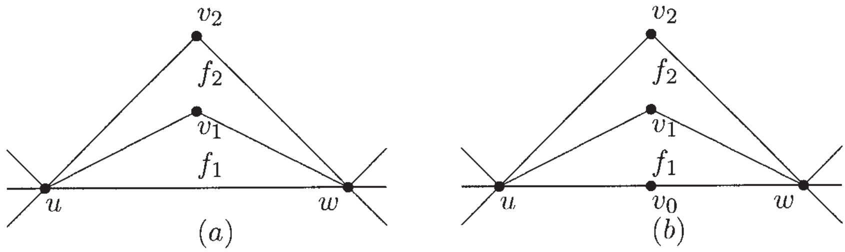

FIGURE 1

Proof. By way of contradiction, we assume that G contains the configuration (a) or (b). Let $G_2 = G \setminus \{v_1, v_2\}$. Then, G_2 remains 2-connected, simple, and $\chi_e(G_2) \le \chi_e(G) \le k$. Thus, G_2 has an edge-face k-coloring $\phi : E(G_2) \cup F(G_2) \to C$. Let f_1' denote the face of G adjacent to f_1 and different from f_2, and let f_2' denote the face adjacent to f_2 and different from f_1. We also use f_1' and f_2' to denote the corresponding faces in G_2. Then, ϕ can be viewed as a partial edge-face k-coloring of all elements of G except $uv_1, uv_2, wv_1, wv_2, f_1, f_2$. Since $f_1, f_2 \in F_s$, it suffices to prove that we can obtain a partial edge-face k-coloring of all elements of G except f_1 and f_2; this can then be extended to the whole of G by Lemma 2.2.

Let $\{a,b\} \subseteq C \setminus \phi(u)$ and $\{c,d\} \subseteq C \setminus \phi(w)$, since $d_{G_2}(u) \le k-2$ and $d_{G_2}(w) \le k-2$. We consider the following two cases.

Case 1. $\phi(f_2') \notin \{a,b\} \cap \{c,d\}$. Without loss of generality, let $\phi(f_2') \notin \{c,d\}$ and $\phi(f_2') \ne a$.

By Lemma 2.1, there exists a one-to-one function $\tau : \{a,b\} \to \{c,d\}$ such that $\tau(a) \ne a$ and $\tau(b) \ne b$. We can color uv_1, wv_1, uv_2, wv_2 with colors $b, \tau(b), a, \tau(a)$, respectively.

Case 2. $\phi(f_2') \in \{a,b\} \cap \{c,d\}$.

Without loss of generality, let $a = c = \phi(f_2')$. If G contains the configuration (a), let $e = \phi(uw)$ and (re)color edges $uw, uv_1, wv_2, uv_2, wv_1$ with colors a, e, e, b, d, respectively. If G contains configuration (b), let $g = \phi(uv_0)$ and note that a, b, g are three different colors; (re)color edges uv_0, wv_1, wv_2 with colors a, a, d, respectively, then color uv_2 with a color in $\{b, g\} \setminus \{d\}$ and color uv_1 with the remaining color from $\{b, g\}$.

In all cases Lemma 2.2 gives an edge-face k-coloring for G, a contradiction. ■

Proposition 3 (Configuration B). *If G contains a configuration illustrated in Figure 2 where f is a face and $d(u) \ge d(w)$ and $d(v) = 2$, then we have the following two conclusions:*

(i) *For each partial edge-face k-coloring* ϕ *of G on* $[E(G) \cup F(G)] \backslash \{uv, wv, f\}$ *which can be obtained from an edge-face k-coloring of* $G \backslash \{v\}$*, let* $e = \phi(uw)$ *and* $a = \phi(f')$ *where* f' *is the face adjacent to* f *and* v*. Then we have that*

(a) $|C \backslash \phi(u)| = 1$ *and consequently,* $d(u) = k$*. And*
(b) *by (a), let* $b \in C \backslash \phi(u)$*. Then* ϕ *must satisfy one of the following two cases: Case* 3.1*. If* $b = a$*, then either* $a \in \phi(w)$ *or* $\{a\} = C \backslash \phi(w)$*, or Case* 3.2*. If* $b \neq a$*, then* $C \backslash \phi(w) \subseteq \{a, b\}$.

(ii)

$$\min\{d(u), d(w)\} \geq 4$$

(*that is,* $d(w) \geq 4$).
(*The vertices u and w are called the terminal vertices of the configuration.*)

Proof. (I) Since $f \in F_s$, it suffices to prove that we can obtain a partial edge-face k-coloring of all elements of G except f; this can then be extended to the whole of G by Lemma 2.2.

(II) If there is a color $d \in C \backslash [\phi(u) \cup \{a\}]$ and a color $c \in C \backslash [\phi(w) \cup \{a\}]$ such that $d \neq c$, then the coloring ϕ can be easily extended to the uncolored edges uv and vw.

(III) Assume that $|C \backslash \phi(u)| \geq 2$. Then $|C \backslash \phi(w)| \geq 2$ because $d(u) \geq d(w)$. By II, $|[[C \backslash \phi(u)] \cup [C \backslash \phi(w)]] \backslash \{a\}| \leq 1$, otherwise there is a pair of colors described in II. Therefore, $|C \backslash \phi(u)| = |C \backslash \phi(w)| = 2$ and $a \in C \backslash \phi(u) = C \backslash \phi(w)$. Let $\{a, c\} = C \backslash \phi(u) = C \backslash \phi(w)$. Let $e = \phi(uw)$. Remove the color e from the edge uw and color it with the color a and then color the edges uv and wv with the colors e, c, respectively. This contradiction implies that $|C \backslash \phi(u)| = 1$ which is (i) (a).

(IV) **Case 3.1.** We assume that $b = a$. If $a \notin \phi(w)$ and $\{a\} \neq C \setminus \phi(w)$, then $a \in C \backslash \phi(w)$ and $|C \backslash \phi(w)| \geq 2$. Let $g \in C \backslash \phi(w)$ and $g \neq a$. Remove the color $e = \phi(uw)$ from the edge uw and color it with the color a and then color the edges uv and wv with the colors e and g, respectively. Therefore, either $a \in \phi(w)$ or $\{a\} = C \backslash \phi(w)$.

(V) **Case 3.2.** We assume that $b \neq a$. Then, $C \backslash \phi(w) \subseteq \{a, b\}$ otherwise there is a color $h \in [C \backslash \phi(w)] \backslash \{a, b\}$ such that $\{h, b\}$ is a pair of colors described in (II).

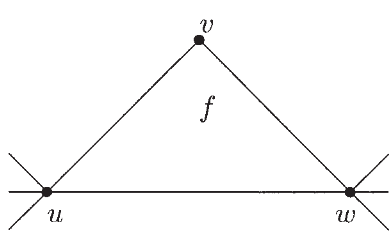

FIGURE 2

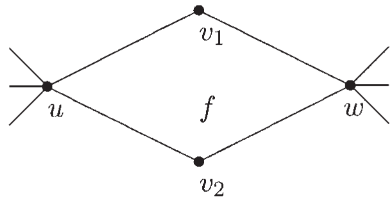

FIGURE 3

(VI) Part (ii) of the proposition can be proved easily by applying the conclusion of (i) (b). If $d(w) \leq 3$, then $a \notin \phi(w)$ and $|C \backslash \phi(w)| = k - (d(w) - 1) \geq k - 2 > 2$. Obviously, it is neither Case 3.1 nor Case 3.2. ∎

Proposition 4 (Configuration C)**.** *If G contains the configuration illustrated in Figure 3, where* $d(v_1) = d(v_2) = 2$ *and f is a face, then*

$$d(u) = d(w) = k.$$

(*The vertices u and w are called the terminal vertices of the configuration.*)

Proof. By way of contradiction, we assume that $d(u) \leq k - 1$. Let $G_4 = G \backslash \{v_1\}$. Then G_4 remains 2-connected and simple and $\chi_e(G_4) \leq \chi_e(G) \leq k$. Hence G_4 has an edge-face k-coloring ϕ. Then ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)] \backslash \{uv_1, wv_1, f\}$. Let f_i' be the face of G, which is adjacent to f and v_i for each i. f_i' is also considered as the corresponding face in G_4. Let $\{a, b\} \subseteq C \backslash \phi(u)$ and $c \in C \backslash \phi(w)$. Let $d = \phi(uv_2)$, $e = \phi(wv_2)$, $g = \phi(f_2')$, and $h = \phi(f_1')$. Since $f \in F_s$, it suffices to prove that we can obtain a partial edge-face k-coloring of all elements of G except f; this can then be extended to the whole of G by Lemma 2.2.

We consider the following two cases:

Case 1. $c \neq h = \phi(f_1')$.

First color the edge wv_1 with color c. If $\{a, b\} \neq \{c, h\}$, then color the edge uv_1 with a color from $\{a, b\} \backslash \{c, h\}$. If $\{a, b\} = \{c, h\}$, then $d \neq c$. Since, in G_4, the faces f_1', f_2' and the edges uv_2, wv_2 are pairwise adjacent to each other, we have $|\{d, e, g, h\}| = 4$. (Re)color the edges uv_1, uv_2 with colors d, h, respectively.

Case 2. $c = h = \phi(f_1')$.

Since $d = \phi(uv_2) \neq h = \phi(f_1') = c$ and $h \neq g = \phi(f_2')$, we remove the color $e = \phi(wv_2)$ from the edge wv_2 and color it with the color h and color the edge wv_1 with the color e. If there is a color in $\{a, b\} \backslash \{e, h\}$, then we can color the edge uv_1 with this color. If $\{a, b\} = \{e, h\}$, (re)color the edges uv_1, uv_2 with colors e, d respectively.

In all cases, Lemma 2.2 gives an edge-face k-coloring for G, a contradiction.

∎

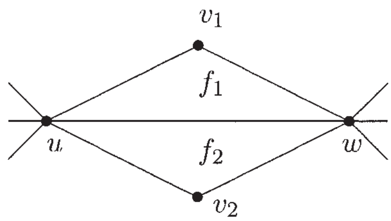

FIGURE 4

Proposition 5 (Configuration D). *Assume that G contains the configuration illustrated in Figure 4, where $d(v_i) = 2$, for each i, and f_1, f_2 are faces. Let f_i' be the face adjacent to f_i and v_i. Let ϕ be a partial edge-face k-coloring of G on $[E(G) \cup F(G)] \backslash \{uv_1, uv_2, wv_1, wv_2, f_1, f_2\}$. Then*
(i)

$$\{a_1, a_2\} = C \backslash \phi(u) = C \backslash \phi(w),$$

where $a_i = \phi(f_i')$ for each i, and consequently,
(ii)

$$d(u) = d(w) = k.$$

***Proof of* (i).** We prove that $\{a_1, a_2\} = C \backslash \phi(u)$; the proof that $\{a_1, a_2\} = C \backslash \phi(w)$ is similar. Note that $|C \backslash \phi(u)| = k - (d_G(u) - 2) \geq 2$. Thus, it is sufficient to prove that $[C \backslash \phi(u)] \backslash \{a_1, a_2\} = \emptyset$. By way of contradiction, we assume that $[C \backslash \phi(u)] \backslash \{a_1, a_2\} \neq \emptyset$. Then, $|[C \backslash \phi(u)] \backslash \{a_1, a_2\}| \geq 1$. Since $f_1, f_2 \in F_s$, it suffices to prove that we can obtain a partial edge-face k-coloring of all elements of G except f_1 and f_2; this can then be extended to the whole of G by Lemma 2.2. We consider the following two cases:

Case 1. $|[C \backslash \phi(u)] \backslash \{a_1, a_2\}| \geq 2$.

Let $\{b_1, b_2\} \subseteq [C \backslash \phi(u)] \backslash \{a_1, a_2\}$ and $\{c_1, c_2\} \subseteq [C \backslash \phi(w)]$. By Lemma 2.1, there is a one-to-one function $\tau_1 : \{a_1, a_2\} \to \{c_1, c_2\}$ such that $\tau_1(a_i) \neq a_i$ for each i. By Lemma 2.1 again, there is a one-to-one function $\tau_2 : \{\tau_1(a_1), \tau_1(a_2)\} \to \{b_1, b_2\}$ such that $\tau_2(\tau_1(a_i)) \neq \tau_1(a_i)$ for each i. Color edges uv_1, uv_2, wv_1, and wv_2 with colors $\tau_2(\tau_1(a_1))$, $\tau_2(\tau_1(a_2))$, $\tau_1(a_1)$, and $\tau_1(a_2)$, respectively.

Case 2. $|[C \backslash \phi(u)] \backslash \{a_1, a_2\}| = 1$.

Note that $|C \backslash \phi(u)| = k - (d_G(u) - 2) \geq 2$. Then either $a_1 \in C \backslash \phi(u)$ or $a_2 \in C \backslash \phi(u)$. Without loss of generality, let $a_1 \in C \backslash \phi(u)$. Let $b \in [C \backslash \phi(u)] \backslash \{a_1, a_2\}$ and $\{c_1, c_2\} \subseteq C \backslash \phi(w)$. By Lemma 2.1, there is a one-to-one function $\tau : \{b, a_2\} \to \{c_1, c_2\}$ such that $\tau(b) \neq b$ and $\tau(a_2) \neq a_2$. Therefore, if $a_1 \notin \{c_1, c_2\}$, color edges uv_1, uv_2, wv_1, and wv_2 with colors b, a_1, $\tau(b)$, and $\tau(a_2)$, respectively. If $a_1 \in \{c_1, c_2\}$, assume that $a_1 = c_1$. Then, $c_2 \neq a_1$. Let $d = \phi(uw)$. Remove the color d from the edge uw and then color it with the color a_1. We further color edges uv_1, uv_2, wv_1, and wv_2 with colors d, b, c_2, and d, respectively.

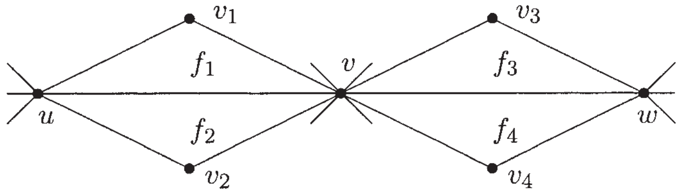

FIGURE 5

In all cases, Lemma 2.2 gives an edge-face k-coloring for G, a contradiction. Therefore, $\{a_1, a_2\} = C\backslash\phi(u)$.

***Proof of* (ii).** By (i), we have $|[C\backslash\phi(u)]| = |[C\backslash\phi(w)]| = 2 = k - (d_G(u) - 2) = k - (d_G(w) - 2)$. Thus, $d_G(u) = d_G(w) = k$. ■

Proposition 6. *G does not contain the configuration illustrated in Figure* 5 *where* $d(v_i) = 2$, $i = 1,\ 2,\ 3,\ 4$ *and* f_i *is a face for each* i.

Proof. By way of contradiction, we assume that G contains this configuration. By Proposition 5, $d_G(u) = d_G(v) = d_G(w) = k$. Let $G_5 = G\backslash\{v_1, v_2, v_3, v_4\}$. Then G_5 remains 2-connected and simple, and $\chi_e(G_5) \leq \chi_e(G) \leq k$. Let $\phi : E(G_5) \cup F(G_5) \rightarrow C$ be an edge-face k-coloring of G_5. The coloring ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)]\backslash \{uv_1, uv_2, vv_1, vv_2, vv_3, vv_4, wv_3, wv_4, f_1, f_2, f_3, f_4\}$. Then $d_{G_5}(v) = k - 4$, $d_{G_5}(u) = d_{G_5}(w) = k - 2$. Thus $|C \setminus \phi(v)| = 4$ and $|C\backslash\phi(u)| = |C\backslash\phi(w)| = 2$. Let $\{a, b, c, d\} = C\backslash\phi(v)$, $\{x, y\} = C\backslash\phi(u)$, and $\{g, h\} = C\backslash\phi(w)$ where $a \notin \{g, h\}$. Let f_i' denote the face adjacent to the vertex v_i and the face f_i for each i.

By Proposition 5 and Lemma 2.2, it is sufficient to extend ϕ to edges uv_1, uv_2, vv_1, vv_2 such that the color set of the remaining two colors for the edges vv_3, vv_4 is not $\{g, h\}$.

(I) We claim that $\{a_1, a_2\} \subset \{b, c, d\}$. Without loss of generality, let $a_1 = c$, $a_2 = d$. Otherwise, without loss of generality, assume that $\{b, c\} \cap \{a_1, a_2\} = \emptyset$. Color the edges uv_1, uv_2 with colors x, y properly. By Lemma 2.1, we may further color the edges vv_1 and vv_2 with the colors b, c properly. Hence, the remaining two colors for the edges vv_3, vv_4 are a and d where $\{a, d\} \neq \{g, h\}$ since $a \notin \{g, h\}$.

(II) If $\{x, y\} \cap \{a_1, a_2\} = \emptyset$, then we may first color the edges vv_1, vv_2, uv_1, uv_2 with colors $a_2(= d), a_1(= c), x, y$, respectively. Then the remaining two colors for the edges vv_3, vv_4 are a and b where $\{a, b\} \neq \{g, h\}$ since $a \notin \{g, h\}$. If $\{x, y\} \cap \{a_1, a_2\} \neq \emptyset$, without loss of generality, let $y = a_2$. Then, $x \neq a_2$. Let $m = \phi(uv)$. Remove the color m from the edge uv and color it with color $a_2(= d)$. Then we color the edges uv_1, uv_2, vv_1, vv_2 with colors m, x, b, m, respectively. Hence, the remaining two colors for the edges vv_3, vv_4 are a and a_1 $(= c)$ where $\{a, a_1\} \neq \{g, h\}$ since $a \notin \{g, h\}$.

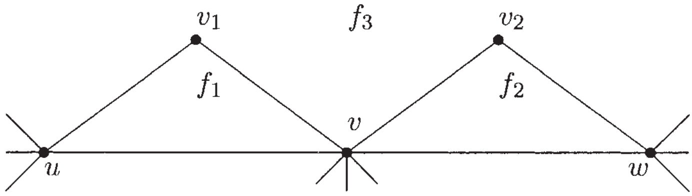

FIGURE 6

In all cases, we obtain a partial edge-face k-coloring ϕ of all elements of G except $vv_3, vv_4, wv_3, wv_4, f_1, f_2, f_3, f_4$ such that $C\backslash\phi(v) \neq C\backslash\phi(w)$. It contradicts Proposition 5. ■

Proposition 7 (Configuration E). *Assume that G contains the configuration illustrated in Figure 6, where $d(v_1) = d(v_2) = 2$, f_1, f_2, f_3 are faces, and the path uv_1vv_2w is on the boundary of f_3. Then,*

$$d(v) \geq k - 2.$$

Proof. By way of contradiction, we assume that $d(v) \leq k - 3$. Let G_6 be the graph obtained from G by removing the two edges vv_1 and vv_2 and adding one edge v_1v_2. Then G_6 remains 2-connected and simple and $|E(G_6)| = |E(G)| - 1 < |E(G)|$. By Theorem 1.2, $\chi_e(G_6) \leq k$. Thus G_6 has an edge-face k-coloring ϕ. Then ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)]\backslash\{vv_1, vv_2, f_1, f_2\}$. Since $d_G(v) + d_G(v_i) \leq k - 3 + 2 = k - 1$ for each i, by Lemma 2.2 ϕ can be extended to the whole of G, a contradiction. ■

Proposition 8. *G does not contain the configuration illustrated in Figure 7, where $d(v) = 4$, $d(v_1) = d(v_2) = 2$, f_1, f_2, f_3 are faces, and f_3 is adjacent to both f_1 and f_2.*

Proof. Consider $G\backslash\{v_1\}$. Then $G\backslash\{v_1\}$ remains 2-connected and thus has an edge-face k-coloring ϕ. The coloring ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)]\backslash\{uv_1, vv_1, f_1\}$ and let $a = \phi(f_3)$. By Proposition 3, $d_G(u) = k$. We may assume that $\phi(vv_2) \neq a$, otherwise we can

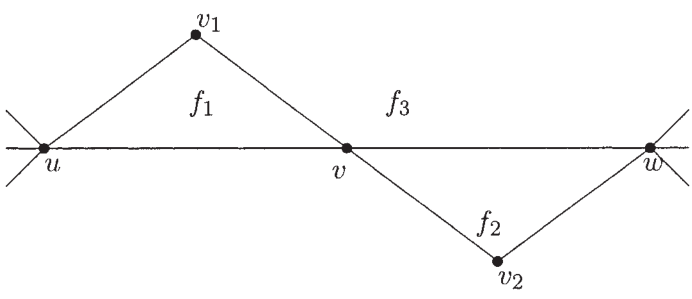

FIGURE 7

replace it with another color from the color set $C\backslash\phi(v)$ since $|C\backslash\phi(v)| = k-(d_G(v)-1) = k-3 \geq 2$. Therefore, it is neither Case 3.1 nor Case 3.2 of Proposition 3 since $d(v) = 4 < k-1$. ■

Proposition 9. *From Proposition* 1, *any subdivided edge of G is of length at most 2. Let $P = uvw$ or uw be a subdivided edge of G of length at most 2 and f be a face in G incident with P. Denote $G\backslash P = G\backslash E(P)$ if P is of length 1 or $G\backslash P = G\backslash\{v\}$ if P is of length 2. Let r_P denote the number of edges $e = xy$ incident with either u or w but not both and with $d(x)+d(y) \leq k-1$. If $d(u)+d(w) \leq r+k$ and $d(f) < \frac{k}{2}$, then $G\backslash P$ has a cut vertex and G/P (contraction) is not simple.*

Proof. Otherwise, by induction hypothesis, G has a partial coloring with the face f and the edges in $E(P)$ uncolored, which is obtained from an edge-face k-coloring of $G\backslash P$ or G/P. By Lemma 2.2, this coloring can be extended to G, a contradiction. ■

4. SERIES–PARALLEL OPERATIONS

A. Contraction of Series–Parallel Subgraphs

The Euler formula (Theorem 5.1 in Section 5) is one of the most useful methods in the study of plane graphs. However, if the minimum vertex degree or the minimum face degree of the graph is 2, the formula may not work effectively. Thus, we have to apply some operations to eliminate subdivided edges and digons of the graph so that the Euler formula may be applied to the resulting graph that is of minimum vertex degree and minimum face degree at least 3.

Operation α: replace each subdivided edge of length at least 2 with a single edge.

Operation β: replace each 2-face with a single edge recursively.

One may repeatedly apply these two operations to G. Since the graph is finite, with a finite number of operations, the resulting graph will be of minimum vertex degree and minimum face degree at least 3 except for the case that G itself is a series parallel graph, in which case the resulting graph is isomorphic to K_2.

The operation sequence is recursively defined as follows:

$$\zeta_1 = \alpha,$$

$$\zeta_{2i} = \beta\,\zeta_{2i-1}, \zeta_{2i+1} = \alpha\,\zeta_{2i}.$$

For any positive integer p, it is obvious that, for each edge $e \in E(\zeta_p(G))$ with endvertices x and y, $\zeta_p^{-1}(e)$ induces a series parallel subgraph in G with terminal vertices x and y.

Let q be the smallest integer such that $\zeta_q(G)$ is of minimum vertex degree and minimum face degree at least 3. Let $H = \zeta_q(G)$. Note that the resulting graph H

is of minimum degree at least 3 and minimum face-degree at least 3. The purpose of applying this sequence of operations on G is to significantly reduce the complexity of the proof, since calculations and reassignments of Euler contributions (defined in Section 5(A)) are much easier for graphs without degree two vertices or degree two faces.

In Section 3, we discover some basic local structures of the minimum counterexample G (called configurations). In this section, we are going to use these configurations to verify that $q = 2$ (Lemma 4.1) and to study the structure of $\zeta_2^{-1}(e)$ in G. In Subsection 4(c)(2), the structure of $\zeta_2(G) = H$ is further studied. By the definition of Euler contribution (see Section 5(A)), the degree sequences of vertices incident to faces with positive Euler contributions are estimated. Then we further prove that each face with positive Euler contribution is adjacent to some faces with negative Euler contribution and with large absolute value. With these results, in Subsection 5(E), we reassign the Euler contributions of the graph H to obtain a new contribution for each face and then we prove that the new contribution of every face is non-positive, which contradicts Theorem 5.1.

B. The Structure of $\zeta_p^{-1}(e)$

With those basic properties in Section 3, we are ready to determine the structure of the subgraph $\zeta_p^{-1}(e)$ in G, for each positive integer p, and each $e \in E(\zeta_p(G))$.

B(1). $p = 1$. It is obvious that $\zeta_1^{-1}(e)$ must be a subdivided edge of length at most 2 (by Proposition 1).

B(2). $p = 2$. For each $e \in E(\zeta_2(G))$ with endvertices u and w, if the multiplicity of e in $\zeta_1(G)$ is at least 2, then $\zeta_2^{-1}(e)$ must be the union of a few subgraphs $J_1, \ldots, J_t$, where $t \geq 2$ is the multiplicity of e in $\zeta_1(G)$, each of which is a subdivided edge of length at most 2 with the endvertices u, w (a Configuration A, by 4(B)(1)). Hence, by Proposition 2, $\zeta_2^{-1}(e)$ must be one of the Configurations B, C, and D described in Propositions 3, 4, and 5

B(3). $p = 3$.

Lemma 4.1. $\zeta_3(G) = \zeta_2(G)$. *That is, operations stop at* $p = 2$.

Proof. It is sufficient to prove that there are no subdivided edges of length at least 2 in $\zeta_2(G)$. By way of contradiction, let $P = u_0u_1 \cdots u_r$ be a subdivided edge of length $r \geq 2$ in $\zeta_2(G)$. Let $e_1 = u_0u_1$ and $e_2 = u_1u_2$. Since $\delta(\zeta_1(G)) \geq 3$, $\zeta_2^{-1}(e_1)$ must be the union of a few subgraphs $J_1, \ldots, J_t$, each of which is a subdivided edge of length at most 2 with the endvertices u_0, u_1, and $\zeta_2^{-1}(e_2)$ must be the union of a few subgraphs $I_1, \ldots, I_s$, each of which is a subdivided edge of length at most 2 with the endvertices u_1 and u_2, where $\max\{s, t\} \geq 2$. We consider the following two cases:

Case 1. Either $t = 1$ or $s = 1$. Without loss of generality, let $s = 1$.

Since $s = 1$, we have $t \geq 2$. By 4(B)(2), $\zeta_2^{-1}(e_1)$ is one of the configurations B, C, and D described in Propositions 3, 4, and 5 with the terminal vertices u_0 and u_1. On the other hand, by Proposition 2, $t \leq 3$. Therefore, $d_G(u_1) \leq 3 + 1 = 4$. If $\zeta_2^{-1}(e_1)$ is Configuration C or D, we must have $d_G(u_1) = k$. Hence, $\zeta_2^{-1}(e_1)$ must be Configuration B. In this case, $t = 2$ and therefore, $d_G(u_1) = 2 + 1 = 3 < 4$. This contradicts (ii) of Proposition 3.

Case 2. Both $t \geq 2$ and $s \geq 2$.

By Proposition 2, we have that $t \leq 3$ and $s \leq 3$. Therefore, $d_G(u_1) \leq 3 + 3 = 6 < k$. Hence, neither $\zeta_2^{-1}(e_1)$ nor $\zeta_2^{-1}(e_2)$ is the Configuration C or D. Thus, both of them must be Configuration B. This implies that $d_G(u_1) = 2 + 2 = 4 \leq k - 3$. By Proposition 8, $\zeta_2^{-1}(u_0u_1u_2)$ must be Configuration E. By Proposition 7, $d_G(u_1) \geq k - 2$, a contradiction. ■

Now, we have proved that $q = 2$, that is, $\zeta_2(G) = \zeta_3(G) = \cdots$. Let $\zeta_2 = \zeta$ and $\zeta(G) = H$.

Let f be a face in G. We say that the face f' in H is the *corresponding face of f* if f' can be obtained from f by replacing subdivided edge of length 2 with single edges in G. In this sense, we also call f to be the *corresponding face of f'*.

C. Some Further Structures of *H*

C(1). Classification of edges of *H*. By the discussion of the previous subsection, we can see that for each edge $e \in H$, $\zeta_2^{-1}(e)$ is one of the Configurations B, C, and D, otherwise $\zeta_2^{-1}(e)$ is either a single edge or a single subdivided edge of length 2. Therefore, the edges of H can be partitioned into three classes:

$$E_1 = \{e \in E(H) : \zeta_2^{-1}(e) \text{ is a subdivided edge of length } 1\},$$

$$E_2 = \{e \in E(H) : \zeta_2^{-1}(e) \text{ is a subdivided edge of length } 2\},$$

$$E_3 = \{e \in E(H) : \zeta_2^{-1}(e) \text{ is one of the Configurations B, C, and D}\}.$$

Obviously,

$$E_1 \subseteq E(G), \quad E_2 \subseteq E(\zeta_1(G)), \quad E_3 \subseteq E(\zeta_2(G)) = E(H).$$

Furthermore, each edge $e \in E_3$ is called a B-edge, a C-edge, or a D-edge if $\zeta^{-1}(e)$ is a B-configuration, a C-configuration, or a D-configuration, respectively; and each edge $e \in E_i$ $(i = 1, 2)$ is called an E_i-edge ($\zeta^{-1}(e)$ is subdivided edge of length i).

C(2). Some structural properties of H.

Lemma 4.2. *The relation between $d_G(v)$ and $d_H(v)$ $(v \in V(H))$ is to be discussed here. We claim that, for each $v \in V(H) \subseteq V(G)$,*

$$d_G(v) \leq 2d_H(v) + 1; \tag{1}$$

$$\textit{if } d_G(v) < k, \quad \textit{then} \quad d_G(v) \leq 2d_H(v); \tag{2}$$

consequently,

$$\textit{if } d_G(v) = k, \quad \textit{then} \quad d_H(v) \geq \frac{k-1}{2}; \tag{3}$$

or, equivalently,

$$\textit{if } d_H(v) < \frac{k-1}{2}, \quad \textit{then} \quad d_G(v) < k. \tag{4}$$

Proof. The degrees of a vertex v would be different in the graphs G and H if v is incident with some B-, C-, or D-edges in H. However, by Proposition 6, no vertex is incident with more than one D-edge. This proves Inequality (1). Furthermore, if $d_G(v), k$, then by Proposition 5, the vertex v is not incident with any D-edge in H. This proves Inequality (2).

Inequalities (3) and (4) are immediate consequences of Inequality (1). ■

Remark. It is obvious that H is loopless, 2-connected and $\delta(H) \geq 3$, and every face is of degree at least 3. Note that the graph H may have some parallel edges, but they do not form degree 2 faces.

Lemma 4.3.

$$\textit{if } e = uv \in E_3, \quad \textit{then} \quad \max\{d_G(u), d_G(v)\} = k, \tag{5}$$

and,

$$\textit{if } \max\{d_H(u), d_H(v)\} < \frac{k-1}{2}, \quad \textit{then } e \in E_1 \cup E_2. \tag{6}$$

Inequality (5) is a corollary of Propositions 3, 4, and 5, since $\zeta^{-1}(e)$ is a Configuration B, C, or D if $e \in E_3$. Inequality (6) is an immediate consequence of Inequalities (4) and (5).

5. CHARGE AND DISCHARGE

A. Euler Contribution

Let M be a plane graph. The *Euler contribution* $\Phi(f)$ of a face f in M is defined as follows:

$$\Phi(f) = 1 - \frac{d(f)}{2} + \sum_{v \in B(f)} \frac{1}{d(v)}$$

where $B(f)$ is the boundary of the face f.

The following theorem by Lebesgue [4] (also see Ore's book [6] Chapter 4, Section 3, page 54) will be applied here to find some special configurations in a plane graph.

Theorem 5.1 (Lebesgue [4]). *Let M be a connected plane graph without loops or bridges. Then*

$$\sum_{f \in F(M)} \Phi(f) = 2.$$

Furthermore, there must be a face with a positive Euler contribution.

A face with positive Euler contribution is called a *positive face*. By Theorem 5.1, H must contain some positive faces.

B. Positive Faces

Lemma 5.2. *By the definition of a positive face, with a simple calculation, a face of H with positive Euler contribution must be in the following list:*

$d_H(f)$	*degree sequence around the face*
5	$3, 3, 3, 3, \le 5$
4	$3, 3, 3, \le \Delta$
4	$3, 3, 4, \le 11$
4	$3, 3, 5, \le 7$
4	$3, 4, 4, \le 5$
3	$5, 6, \le 7$
3	$5, 5, \le 9$
3	$4, 7, \le 9$
3	$4, 6, \le 11$
3	$4, 5, \le 19$
3	$4, 4, \le \Delta$
3	$3, 11, \le 13$
3	$3, 10, \le 14$
3	$3, 9, \le 17$
3	$3, 8, \le 23$
3	$3, 7, \le 41$
3	$3, \le 6, \le \Delta$

With further investigation, we will prove that the length of a positive face in H is exactly 3 and the maximum degree of the vertices on its boundary is very large (See Lemma 5.3).

Lemma 5.3. (i) *H does not contain a 4-face $x_1x_2x_3x_4x_1$ such that $d_H(x_1) = 3$, $d_H(x_2) \leq 11$, and $d_H(x_4) \leq 11$.*

(ii) *Let $f = x_1x_2\cdots x_dx_1$ be a positive face, then*

$$d = 3;$$

(iii)

$$\max\{d_H(x_1), \ldots, d_H(x_3)\} \geq 12;$$

(iv) Let $d_H(x_3) \geq d_H(x_2) \geq d_H(x_1)$. Then, $d_H(x_2) \leq 11 < k - 1/2$, $d_H(x_1) \leq 4 < k - 1/2$, and $d_H(x_1) + d_H(x_2) \leq 14$.

Proof. (i) Assume that H contains a 4-face $f = x_1x_2x_3x_4x_1$ with $d_H(x_1) = 3$, $d_H(x_2) \leq 11$, and $d_H(x_4) \leq 11$. Let f' be the corresponding face of f in G. By Lemma 4.3, $E(\zeta_2^{-1}(x_1x_i))$ is a subdivided edge of length at most 2 for each $i = 2, 4$. Note that $d_G(x_1) + d_G(x_i) \leq 2 \times 10 + 1 + 2 + 2 \times 1 = 25$ with equality if x_1 is incident with a B-edge, that is, in G, x_1 is adjacent to a 2-vertex. Therefore, by Proposition 9, $G\backslash\zeta^{-1}(x_1x_i)$ has a cut vertex and $G/\zeta^{-1}(x_1x_i)$ is not simple for each $i = 2, 4$. Then, $x_1x_3 \in E(H)$ and in $H\backslash x_1x_i$, x_3 is a cut vertex. Furthermore, $\{x_1, x_3\}$ is a 2-cut of H and $H\backslash\{x_1x_2, x_1x_4\}$ has three blocks, one of which is the edge x_3x_1 (see Fig. 8).

Let Q_1, Q_2 be the two components of $H - \{x_1, x_3\}$ and let $H_i = H - V(Q_i)$ for each $i = 1, 2$. Hence $H_1 \cap H_2 = \{x_1, x_3, x_1x_3\}$. Since the edge x_1x_3 appears in both H_1 and H_2, both H_1 and H_2 are simple and 2-connected. Let f_1 be the face in H_1 adjacent to the triangle $x_3x_2x_1x_3$ and the edge x_1x_3, and let f_2 be the face in H_2 adjacent to the edges x_1x_3, x_1x_4, and the face $x_3x_4x_1x_3$ (see Fig. 8).

Let G_i be the subgraph of G corresponding to H_i for each $i = 1, 2$. We also denote by f_1', f_2' the corresponding faces of f_1, f_2 in G_1, G_2, respectively. Then both G_1 and G_2 remain simple and 2-connected and therefore, by induction hypothesis, G_i has an edge-face k-coloring ϕ_i for each $i = 1, 2$.

Assume that $x_3x_1 \in E(G)$. That is, $x_1x_3 \in E_1$. For other cases ($x_3x_1 \in E_2$ *or* E_3), the arguments are similar. We relabel the colors such that $\phi_1(x_3x_1) = \phi_2(x_3x_1) = a$ and $\phi_1(x_3) \cap \phi_2(x_3) = \{a\}$. Let $\phi_1(f_1') = \alpha$ and

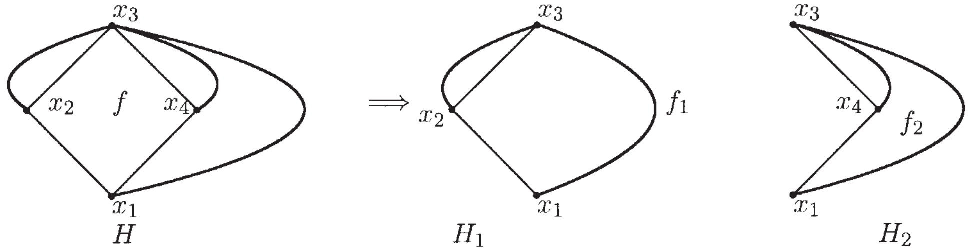

FIGURE 8

$\phi_2(f_2') = \beta$. If $\alpha = \beta$, without loss of generality, let $\alpha \notin \phi_1(x_3)\backslash\{a\}$. We relabel the colors in $[\phi_1(x_3) \cup \{\alpha\}]\backslash\{a\}$ in the coloring ϕ_1 of G_1 such that $\alpha \neq \beta$. Hence, without loss of generality, let $\alpha \neq \beta$. Now we uncolor the faces corresponding to the triangle $x_1x_3x_2x_1$ in H_1 and the triangle $x_1x_3x_4x_1$ in H_2. Then, after gluing G_1 and G_2 together, we can get a partial edge-face k-coloring ϕ of G with the edges in $E(\zeta_2^{-1}(x_1x_4))$ and the face f' uncolored. It is easy to see that these uncolored elements are in $E_s \cup F_s$ (defined before Lemma 2.2). Therefore, by Lemma 2.2, ϕ can be extended to the graph G, a contradiction.

(ii) By (i) and the table in Lemma 5.2, either $d = 5$ or $d = 3$. Assume that $d = 5$. Without loss of generality, let $d_H(x_i) = 3$ for each $i = 1, 2, 3, 4$.

Let u be a vertex adjacent to x_2 other than x_1 and x_3. Notice that x_1 and x_3 are not adjacent in H, otherwise, x_2 is a cut vertex separating u from x_5. Similarly, x_2 and x_4 are not adjacent in H. Since $d_G(x_2) + d_G(x_3) \leq 2 \times (3 + 3) = 12 \leq k$, by Proposition 9, $H\backslash\{x_2x_3\}$ has a cut vertex and H/x_2x_3 is not simple and therefore, x_5 is adjacent to both x_2 and x_3. This implies that x_5 is a cut vertex in $H\backslash\{x_2x_3\}$ and x_3, x_1 are in different blocks.

Let H' be the graph obtained from $H\backslash\{x_2x_3\}$ by adding the edge x_1x_3 to $H\backslash\{x_2x_3\}$ such that $x_5x_1x_3x_4x_5$ is a 4-face in H'. Then H' is 2-connected. Let G' be the graph corresponding to H'. Then $|E(G')| = |E(G)|$. Assume that G' is also a smallest counterexample to Theorem 1.3. Notice that H' contains a 4-face $x_1x_3x_4x_5x_1$ with $d_{H'}(x_1) = d_{H'}(x_3) = d_{H'}(x_4) = 3$. It contradicts Lemma 5.3 (i). So, G' is not a counterexample. That is, G' has an edge-face k-coloring ϕ, which can be easily adjusted to be an edge-face k-coloring of G, a contradiction. Therefore, $d = 3$.

(iii) We may assume that $d_H(x_1) \leq d_H(x_2) \leq d_H(x_3)$. By way of contradiction, we assume that $d_H(x_3) \leq 11$. Then, $d_H(x_i) < (k - 1)/2$ and, by Inequality (4) in Lemma 4.2, we have $d_G(x_i) < k$ for each $i = 1, 2, 3$. Therefore, in H, no C-edges or D-edges is incident with the vertex x_i and, by Inequality (6) in Lemma 4.3, $x_ix_{i+1} \in E_1 \cup E_2$ for each $i = 1, 2, 3$.

Let m_i denote the number of B-edges incident with the vertex x_i in H. Then $d_G(x_i) = d_H(x_i) + m_i$. It is obvious that either $G\backslash E(\zeta_2^{-1}(x_1x_2))$ is 2-connected and simple or $G\backslash E(\zeta_2^{-1}(x_1x_3))$ is 2-connected and simple. Without loss of generality, let $G\backslash E(\zeta_2^{-1}(x_1x_3))$ be 2-connected and simple. Let ϕ be an edge-face k-coloring of $G\backslash E(\zeta_2^{-1}(x_1x_3))$. Remove the colors from those edges, which have endvertices x_i and a 2-vertex for each $i = 1, 3$. Notice that there are at least m_i 2-vertices adjacent to x_i for each $i = 1, 2, 3$. If $\zeta_2^{-1}(x_1x_3) = x_1x_3$, there are at least $k - (d_G(x_1) + d_G(x_3) - 2) + m_1 + m_3 - 1 = k - (d_H(x_1) + m_1 + d_H(x_3) + m_3 - 2) + m_1 + m_2 - 1 = k - (d_H(x_1) + d_H(x_3)) + 1 \geq 24 - 14 + 1 = 11$ colors available for the edge x_1x_3. If $\zeta_2^{-1}(x_1x_3) = x_1x_0x_3$ where $d_G(x_0) = 2$. Then ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)]\backslash\{x_1x_0, x_0x_3, f\}$. Notice that the uncolored face f is the corresponding face of f' whose length is at most $3 \times 2 = 6$ and the uncolored edges are in E_s (defined above Lemma 2.2). Therefore, by Lemma 2.2, ϕ can be extended to the graph G, a contradiction.

(iv) is obvious by the table in Lemma 5.2. ■

Let $f' = uvwu$ be a positive face in H with $d_H(u) \leq d_H(v) \leq d_H(w)$. The edge uv is called *special* and the face in H incident with the special edge uv other than the face f' is also called *special* with respect to the edge uv.

C. Strategy of the Remaining Part of the Proof

We will re-assign the Euler contribution of the graph H (or, commonly called *charge/discharge*) in Subsection 5(E) as follows: The Euler contribution of every positive face will be discharged to a special face by crossing a special edge. Consequently, we will show that after re-assignment, H will have no face with positive charge. It is obvious that the new charges of the non-special faces are non-positive. We will prove that the new charge of each special face remains non-positive. Notice that each special face receives some charge from adjacent positive faces sharing special edges.

In order to keep the new charge of a special face non-positive, it is sufficient to prove that the initial charge of a special face is negative and that the absolute value of its initial charge is very large. By Theorem 5.1, the initial charge (Euler contribution) of a face is determined by its length and the degrees of the vertices on its boundary. Therefore, it is sufficient to prove that the length of each special face is large enough (see Lemma 5.5) and that there are enough number of vertices with large degrees (see Lemma 5.6).

D. Lemmas for Charge and Discharge

Let $SPE(H)$ denote the set of all special edges of H and $SPE_1(H)$, and the set of all such special edges both of whose endvertices are of degree 3 in H. Let $SPE_2(H) = SPE(H) \backslash SPE_1(H)$.

Lemma 5.4. *For each special edge $uv = e \in SPE(H)$ with uvw as the adjacent positive face, we have*

(i) $e \in E_1 \cup E_2$;

(ii) *For any $A \subseteq E(G) \cup F(G)$, any partial edge-face k-coloring ϕ of G on A can be adjusted and then extended to $A \cup \zeta^{-1}(e)$;*

(iii) *$G \backslash E(\zeta^{-1}(e))$ is not 2-connected, and the vertex w is the cut-vertex of the graph $G \backslash E(\zeta_2^{-1}(e))$;*

(iv) $e \in E_1$.

Proof. (i) It is obvious by Inequality (6) in Lemma 4.3, and Lemma 5.3 (iv) that $e \in E_1 \cup E_2$.

(ii) It is sufficient to show that for each $e = uv \in Q'$, the coloring ϕ can be adjusted and then, extended to the edges in $E(\zeta^{-1}(e))$. By Lemma 5.3 (iv), $d_H(u) \leq d_H(v) \leq 11 < (k-1)/2$. Therefore, by Inequality (4) in Lemma 4.2, $\max\{d_G(u), d_G(v)\} < k$. Thus, the vertices u, v are not incident with any C- or D-edges by Propositions 4 and 5.

Let m_1 denote the number of B-edges incident with u and m_2 the number of B-edges incident with v. Then $d_G(u) = d_H(u) + m_1$ and $d_G(v) = d_H(v) + m_2$. Let $E' =$ the set of edges in G with endvertices u and a 2-vertex and $E'' =$ the set of edges in G with endvertices v and a 2-vertex. Notice that, by Lemma 5.3 (iv), $d_H(u) + d_H(v) \leq 14$.

Remove the colors from the edges in $E' \cup E''$ and then color the edges in $\zeta^{-1}(e)$ since there are at most $d_H(u) - 1 + d_H(v) - 1 + 2 \leq 14$ forbidden colors for each of those edges. Since $e = uv \in E_1 \cup E_2$, there are at most $d_H(u) - 1$ B-edges incident with the vertex u. Therefore, $d_G(u) = d_H(u) + m_1 \leq d_H(u) + d_H(u) - 1 \leq 11 + 11 - 1 = 21$ since $d_H(u) \leq 11$. Similarly, $d_G(v) = d_H(v) + m_2 \leq 21$. Therefore, for each edge xy in $E' \cup E''$, $d_G(x) + d_G(y) \leq 2 + 21 = 23 \leq k - 1$. By Lemma 2.2, we can recolor the edges in $E' \cup E''$.

(iii) If $G \backslash \zeta^{-1}(e)$ is 2-connected, then it has an edge-face k-coloring ϕ. By (ii), the coloring ϕ can be adjusted and then extended to the edges of $\zeta^{-1}(e)$. Since $d_G(uvw) \leq 6$, by Lemma 2.2, the coloring ϕ can be further extended to the positive face $uvwu$ and therefore the entire G.

(iv) By (i), assume that $e \in E_2$. By (iii), since $G \backslash \zeta^{-1}(e)$ has a cut-vertex w, it is impossible that $G \backslash \zeta^{-1}(e)$ has an edge joining u and v. By Inequality (1) in Lemma 4.2 and Lemma 5.3 (iv), one of $d_G(u)$ and $d_G(v)$ is at most $2 \times 4 + 1 = 9 < k - 2$. It contradicts Proposition 1 (ii) that the degree of each of $\{u, v\}$ must be at least $k - 2$ in G. ■

Lemma 5.5. *For any special face f'', let f be its corresponding face in G. Let s be the number of special edges in the boundary of f''. Then,*

$$2d_H(f'') \geq d_G(f) + s; \tag{7}$$

$$d_G(f) \geq \frac{k}{2} + s; \tag{8}$$

$$d_H(f'') \geq \frac{k}{4} + s. \tag{9}$$

Proof. (7) Let $e \in E(f'')$ in H. If $\zeta^{-1}(e)$ is not an edge in G, then the subgraph of G induced by $\zeta^{-1}(e)$ must be an E_i-, B-, C-, or D-edge. Thus, the edge e in H corresponds to a subdivided edge of length 1 or 2 around the boundary of f in G. By Lemma 5.4 (iv), every special edge is an original edge in G. Therefore,

$$d_G(f) \leq 2d_H(f'') - s.$$

(8) Let uv be a special edge incident with f''. By Lemma 5.4 (iii), $G \backslash \{uv\}$ is not 2-connected. Let $f' = vuw$ be the positive face adjacent to the special edge $e = uv$ in H. By Lemma 5.4 (iii), w is a cut vertex in $G \backslash \{e\}$. Moreover, w separates $G \backslash \{e\}$ into two blocks, say, G' and G'', and each block is 2-connected and they share the face f and the vertex w. Thus, G' and G'' both have edge-face

k-colorings ϕ' and ϕ'' such that $\phi'(w) \cap \phi''(w) = \emptyset$. Let F' be the set of all faces of G adjacent to the face f'' in H whose corresponding faces in H are positive, and E' the set of all special edges incident with the face f. We remove the colors from the faces and edges of $E' \cup F' \cup \{f\}$. Then we can combine the colorings ϕ' and ϕ'' into a partial edge-face-coloring ϕ of G: $\phi : [E(G) \cup F(G)] \setminus [E' \cup F' \cup \{f\}] \to C$.

If the partial coloring ϕ can be extended to the special face f, we can further color the edges in E' by Lemma 5.4 (ii) and the faces in F' by Lemma 2.2 since by Lemma 5.3, the length of each positive face is at most $2 \times 3 = 6 < (k-1)/2$. So, the partial coloring ϕ cannot be extended to the face f.

Obviously,

$$|E' \cup F'| \geq 2s.$$

Thus, there are at most

$$2d_G(f) - |E' \cup F'| \leq 2d_G(f) - 2s$$

forbidden colors for the face f.

Assume that

$$2d_G(f) - 2s \leq k - 1.$$

Then there are at least $k - (2d_G(f) - 2s) \geq 1$ colors available for the face f. Therefore, ϕ can be extended to the face f, a contradiction. Hence, we must have

$$2d_G(f) - 2s \geq k.$$

(9) By Inequalities (7) and (8), we have

$$2d_H(f'') \geq d_G(f) + s \geq \left[\frac{k}{2} + s\right] + s.$$

Hence,

$$d_H(f'') \geq \frac{k}{4} + s.$$ ■

Lemma 5.6. *For each special edge $uv \in SPE_1(H)$, let $f' = uvw$ be the positive face adjacent to the edge uv. Let u_1 be the vertex in H adjacent to u other than v and w, and v_1 be the vertex in H adjacent to v other than u and w. Then,*

$$\max\{d_H(u_1), d_H(v_1)\} \geq \frac{k-4}{2}.$$

Proof. Notice that $d_H(u) = d_H(v) = 3$ since $uv \in SPE_1(H)$.

(I) By way of contradiction, we assume that both $d_H(u_1) < (k-4)/2$ and $d_H(v_1) < (k-4)/2$. Then, by Inequality (6) in Lemma 4.3, the edges uu_1 and vv_1 are all in $E_1 \cup E_2$ and by Lemma 5.4 (iv), $uv \in E_1$. Let $uu'_1 \in E(\zeta^{-1}(uu_1))$ and $vv'_1 \in E(\zeta^{-1}(vv_1))$. Note that either $u'_1 = u_1$ or $d_G(u'_1) = 2$ and either $v'_1 = v_1$ or $d_G(v'_1) = 2$. Let f'' denote the face in H adjacent to the face f' and incident with the edge uw. Let $f_1,\ f_2$ be the corresponding faces of f' and f'' in G, respectively.

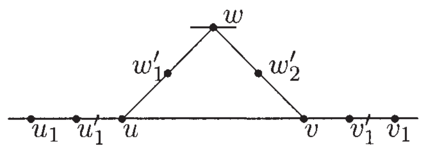

FIGURE 9

(II) Claim that both $uw \in E_1 \cup E_2$ and $vw \in E_1 \cup E_2$.

By way of contradiction, we assume that $uw \in E_3$. Since $d_H(u) \leq 11 < (k-1)/2$, by Inequality (4) in Lemma 4.2, $d_G(u) < k$. Therefore, $\zeta^{-1}(uw)$ must be a B-edge. Let w_3 be the only 2-vertex in $\zeta^{-1}(uw)$. If w_3 is on the boundary of f_1. Then, ww_3 is adjacent to two faces with length at most 6 and $d_G(w_3) + d_G(u) \leq 2 + 2 \times 11 = 24 \leq k$ and uw_3 is also adjacent to two faces with length at most 6. Therefore, by Lemma 2.2, any edge-face k-coloring of the graph $G\backslash\{w_3\}$ can be extended to the graph G. Therefore, w_3 must be on the boundary of f_2.

Clearly, $G\backslash\{uw\}$ remains 2-connected and simple. Let ϕ be an edge-face k-coloring of $G\backslash\{uw\}$. Remove the colors from the edges uu_1', uv, and uw_3 and from the face f_1. Denote $S = \{uw, uu_1, uv, uw_3, f_1, uw_3wu\}$. Then ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)]\backslash S$. Obviously, there is at least one color available for the edge uw and color it. Since $S\backslash\{uw\}$ is a subset of $E_s \cup F_s$ defined above Lemma 2.2, by Lemma 2.2 ϕ can be adjusted and then extended to G. This contradiction shows that $uw \in E_1 \cup E_2$. Similarly, we can also prove that $vw \in E_1 \cup E_2$.

(III) Note that $u_1u, uw, uv \in E_1 \cup E_2$. We have $d_G(u) = d_H(u) = 3$. Similarly, $d_G(v) = d_H(v) = 3$.

(IV) Let $ww_1' \in E(\zeta^{-1}(uw))$ and $ww_2' \in E(\zeta^{-1}(vw))$. Note that either $w_1' = u$ or $d_G(w_1') = 2$, and that either $w_2' = v$ or $d_G(w_2') = 2$. By Lemma 5.4 (iii), $G\backslash E(\zeta_2^{-1}(uv))$ is not 2-connected. Therefore, $G\backslash E(\zeta_2^{-1}(uw))$ remains 2-connected and simple. Let ϕ be an edge-face k-coloring of the graph $G\backslash E(\zeta_2^{-1}(uw))$. Remove the colors from the edges uu_1', uv, vv_1', and vw_2' (if any). Then, ϕ can be viewed as a partial edge-face k-coloring of G with the elements $ww_1', uw_1', uu_1', uv, vv_1'$, vw_2' (if any), and f_1, uncolored.

Let $a = \phi(ww_2')$, $b = \phi(f_2)$, and $c \in C\backslash\phi(w)$. If $c \neq b$, we can color the edge ww_1' with the color c. If $c = b$, remove the color a from the edge ww_2' and then color it with the color b and then color the edge ww_1' with the color a. The remaining uncolored elements are the edges uu_1', vv_1', uw_1' (if any), and vw_2' (if any) and the face f_1. Notice that these elements are all in $E_s \cup F_s$ defined above Lemma 2.2. Therefore, ϕ can be extended to the graph G, a contradiction.

E. Charge and Discharge—The Final Step

Consider Φ, the Euler contribution of H, as the initial charge of the face set of H. We will reassign a new charge Φ' to each face of H as follows. Each positive face f' sends its total amount of its Euler contribution $\Phi(f')$ to the adjacent special face sharing the special edge with it by crossing the special edge.

We now check the new charge $\Phi'(f')$.

(a) For each non-special face f^* with $\Phi(f^*) \leq 0$, the charge remains the same. That is,

$$\Phi'(f^*) = \Phi(f^*) \leq 0.$$

(b) For each positive face f' in H,

$$\Phi'(f') = \Phi(f') - \Phi(f') = 0,$$

and if the positive face f' is adjacent to a special edge in SPE_1, then

$$\Phi(f') \leq 1 - \frac{3}{2} + 2 \times \frac{1}{3} + \frac{1}{12} = \frac{1}{4}.$$

If the positive face f' is adjacent to a special edge in SPE_2, then

$$\Phi(f') \leq 1 - \frac{3}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{12} = \frac{1}{6}.$$

In summary, each positive face in H discharges either $\leq 1/6$ or $\leq 1/4$ to an adjacent special face sharing the special edge with it by crossing a special edge $e \in SPE_2$ or $e \in SPE_1$, respectively.

(c) For each special face f', Let $r = d_H(f')$ and s_i be the number of special edges in $SPE_i(H)$ adjacent to f' for each $i = 1, 2$.

(d) By Lemma 5.6, there are at least $s_1/2$ vertices in $B(f')$ with degrees at least $(k-4)/2$.

(e) By Inequality (9) in Lemma 5.5, we have

$$r \geq \frac{k}{4} + (s_1 + s_2). \tag{10}$$

Therefore,

$$\begin{aligned}
\Phi'(f') &\leq \Phi(f') + \frac{s_1}{4} + \frac{s_2}{6} \\
&= 1 - \frac{r}{2} + \sum_{v \in B_H(f')} \frac{1}{d_H(v)} + \frac{s_1}{4} + \frac{s_2}{6} \\
&\leq 1 - \frac{r}{2} + \left[\frac{s_1}{2} \times \frac{2}{k-4} + \left(r - \frac{s_1}{2}\right) \times \frac{1}{3}\right] + \frac{s_1}{4} + \frac{s_2}{6} \quad (\text{by } (d)) \\
&= 1 - \frac{r}{6} + \frac{s_1}{k-4} + \frac{s_1}{12} + \frac{s_2}{6} \\
&\leq 1 - \left[\frac{k}{24} + \frac{s_1 + s_2}{6}\right] + \frac{s_1}{12} + \frac{s_2}{6} + \frac{s_1}{k-4} \quad (\text{by } (e)) \\
&= 1 - \frac{k}{24} - \frac{s_1}{12} + \frac{s_1}{k-4} \\
&\leq 0 \ (\text{since } k \geq 24).
\end{aligned}$$

Thus,

$$2 = \sum_{f' \in F(H)} \Phi(f') = \sum_{f' \in F(H)} \Phi'(f') \leq 0$$

This contradiction completes the proof of Theorem 1.4. ■

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