Chords of Longest Circuits of Graphs Embedded in Torus and Klein Bottle

Xuechao Li[1] and Cun-Quan Zhang[2]*

[1]DIVISION OF ACADEMIC ENHANCEMENT
THE UNIVERSITY OF GEORGIA
ATHENS, GEORGIA 30602-5554
E-mail: xcli@uga.edu

[2]DEPARTMENT OF MATHEMATICS
P.O.BOX 6310
WEST VIRGINIA UNIVERSITY
MORGANTOWN, WEST VIRGINIA 26506-6310
E-mail: cqzhang@math.wvu.edu

Received December 15, 2000; Revised September 17, 2002

DOI 10.1002/jgt.10091

Abstract: Thomassen conjectured that every longest circuit of a 3-connected graph has a chord. It is proved in this paper that every longest circuit of a 4-connected graph embedded in a torus or Klein bottle has a chord. © 2003 Wiley Periodicals, Inc. J Graph Theory 43: 1–23, 2003

Contract grant sponsor: National Security Agency; Contract grant numbers: MDA904-00-1-00614, MDA904-01-1-0022.
*Correspondence to: Xuechao Li, Division of Academic Enhancement, The University of Georgia, Athens, GA 30602-5554. E-mail: xcliauga.edu

Keywords: *longest circuit; chords; embedded graphs; charge and discharge*

1. INTRODUCTION AND TERMINOLOGY

Thomassen conjectured ([1]) that every longest circuit of a 3-connected graph has a chord. In 1987, C. Q. Zhang [9] proved that every longest circuit of a 3-connected planar graph G has a chord if G is cubic or $\delta(G) \geq 4$. In 1997, Carsten Thomassen [8] proved that every longest circuit in a 3-connected cubic graph has a chord. We shall here prove the following theorem.

Theorem 1.1 (The Main Theorem). *Every longest circuit of a* 4*-connected graph embedded in the torus or the Klein bottle has a chord.*

The method that we shall use in the proof of Theorem 1.1 is very different from those in papers [8] and [9]. Methods used in those papers are based on connectivity, enumeration, Hamilton circuits, and vertex coloring. Here, we use Euler contribution, charge/discharge method.

Throughout this paper, we consider only finite simple graphs. For a graph G, the vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively.

Let u, $v \in V(G)$. The vertex u is a *neighbor* of v if $uv \in E(G)$. The set of all neighbors of v is denoted by $N(v)$ and the set of edges incident with the vertex v is denoted by $E(v)$. The *degree of a vertex* v, denoted by $d_G(v)$ (or simply, $d(v)$, if there is no confusion), is the number of neighbors of v. For a subgraph P of G, $d_P(v) = |N(v) \cap V(P)|$.

Let P be a subgraph of a graph G. An edge e is *a chord* of P if e is not an edge of P and both end vertices of e are in P. A *P-bridge* of G is either a chord of P or a subgraph of G induced by the edges in a component of $G\backslash V(P)$ and all edges that join the component and P. For a P-bridge B of G, the vertices in $B \cap P$ are the *attachments of* B (*on* P) denoted by A(B), and $I(B)$ is the set of vertices of the bridge B (excluding the attachment vertices on P).

The *length* of the boundary of a face f (or, simply, the *length* of f, the *degree* of f) of a graph G is denoted by $d_G(f)$. The set of edges incident with a face f in a graph G is denoted by $E_G(f)$, and the set of vertices incident with a face f in a graph G is denoted by $V_G(f)$.

Let G be a 4-connected graph, $C = v_1v_2 \cdots v_iv_{i+1} \cdots v_mv_1$ be a longest circuit of G. Let $v_i, v_j \in V(C)$ with $i < j$, the segment $v_iv_{i+1} \cdots v_{j-1}v_j$ of C is denoted by v_iCv_j, the segment $v_jv_{j-1} \cdots v_{i+1}v_i$ of C is denoted by $v_j\bar{C}v_i$.

Let G be embedded in a surface S. If each face of G is isomorphic to an open disk, then this embedding is called an *open* 2*-cell embedding*. Note that every graph has an open 2-cell embedding in some surface and all embeddings we considered in this paper are open 2-cell embeddings.

For the sake of convenience in the later discussion, if $e = xy$ and f', f'' are faces on the two sides of the edge e, we say that *the edge e is associated with sequence* $\{x, y, f', f''\}$. (One should be careful that it is possible that f' may be the same as f'' in some cases.)

2. EULER CONTRIBUTION

Let a graph H be embedded in a surface. For a vertex $v \in V(H)$, let $\{e_1, \dots, e_{d(v)}\} = E(v)$ where e_i, e_{i+1} are on the boundary of a face (where $e_{d(v)+1} = e_1$). *An angle at v of G* is a pair of edges $\{e_i, e_{i+1}\}$ (where $e_{d(v)+1} = e_1$).

Denote the set of all angles of H by Λ. For an angle $\alpha \in \Lambda$ at a vertex v and at corner of a face f, denote the vertex v by v_α and face f by f_α. Note that there are $d(v)$ angles at vertex v and there are $d(f)$ angles at the corners of a face f and each edge appears in four angles and each angle consists of two edges

$$|V(H)| = \sum_{\alpha \in \Lambda(H)} \frac{1}{d(v_\alpha)}, \ |E(H)| = \sum_{\alpha \in \Lambda(H)} \frac{1}{2}, \ |F(H)| = \sum_{\alpha \in \Lambda(H)} \frac{1}{d(f_\alpha)}.$$

By the following Euler formula for the torus and the Klein bottle

$$|V(H)| - |E(H)| + |F(H)| = 0,$$

we have the *Lebesgue's formula* ([3])

$$\sum_{\alpha \in \Lambda(H)} \left[\frac{1}{d(v_\alpha)} + \frac{1}{d(f_\alpha)} - \frac{1}{2}\right] = 0. \tag{1}$$

For each angle α, the general term of (1)

$$\Phi(\alpha) = \frac{1}{d(v_\alpha)} + \frac{1}{d(f_\alpha)} - \frac{1}{2} \tag{2}$$

is called the *Euler contribution of the angle α*.

For an edge $e = v_1 v_2$, let f_1, f_2 be the faces incident with e. Note that e appears in four angles and each angle consists of two edges. When one sums a half of the Euler contribution of all angles containing e, one obtains the *Euler contribution of the edge e*

$$\Phi(e) = \frac{1}{d(v_1)} + \frac{1}{d(v_2)} + \frac{1}{d(f_1)} + \frac{1}{d(f_2)} - 1. \tag{3}$$

According to Lebesgue's formula (1), we have the total Euler contributions of all angles and all edges

$$\sum_{\alpha \in \Lambda(H)} \Phi(\alpha) = \sum_{e \in E(H)} \Phi(e) = 0. \tag{4}$$

3. LEMMAS AND DEFINITIONS

The application of Euler contribution and the search of edges with positive Euler contributions will lead our attention to the local structures of some adjacent pairs of small faces. Though the global embedding in the torus or the Klein bottle is different from that in the sphere, the local structure of a subgraph embedded in an open disk neighborhood of the torus or the Klein bottle may appear very similar to the planar graph. In order to avoid any possible misuse of properties that are for sphere but not for the torus or the Klein bottle, we need two lemmas (Lemma 3.1 and Lemma 3.2), which describe some local properties of faces and its boundaries in the torus and the Klein bottle. Lemma 3.1 (together with some claims, such as claim (5) in the proof) enables us to work on some small faces locally without worrying about any complicated structure around the boundaries (very much like in the sphere).

Remark. Note that the embedding we are talking about here is an open 2-cell embedding. That is, each face is isomorphic to an open disk. However, one should not confuse it with closed 2-cell embedding, in which, the closure of every face is isomorphic to a closed disk. Therefore, we should not assume that the boundary of any face is a circuit since it is possible that some edge might be passed through twice by the boundary of a face.

Definition 3.1. *Let H be a graph embedded in a surface S. A face f of H is good if no edge of H is passed by the boundary of f more than once. A face f is bad if otherwise.*

Lemmma 3.1. *Let H be a connected triangle-free graph with $\delta(H) \geq 2$ which has an open 2-cell embedding in a surface S. Let f be a face of G in S. If f is a bad face of G, then the length of f must be at least 10 if the surface S is orientable, and is at least 8 if S is non-orientable.*

Proof. Let $C = v_1 \cdots v_r v_1$ be the boundary of the face f. Let the closed walk C be oriented in the order $C = v_1 \cdots v_r v_1$. Since f is bad, C is not a circuit. Hence, some edge e is passed twice in C.

Case 1. The edge e is passed twice in opposite directions. Assume that $e = xy$ is $v_1 v_2 = xy$ and $v_{\mu-1} v_\mu = yx$ in the closed walk C. Since $\delta(H) \geq 2$, the graph induced by the closed subwalk $v_2 \ldots v_{\mu-1}$ contains a circuit Q_1 of length at most

$\mu - 3$. Furthermore, Q_1 is of length at least 4, since the graph H is triangle-free. Symmetrically, the graph induced by the closed subwalk $v_\mu \cdots v_r v_1$ contains a circuit Q_2 of length at most $r - \mu + 1$, which is also at least 4. So, it is obvious that $r \geq 10$.

Case 2. The edge e is passed twice in the same direction. (Note that, this case occurs only in non-orientable surface.) Assume that $e = xy$ is $v_1 v_2 = xy$ and $v_{\mu-1} v_\mu = xy$ in the closed walk C. Since $\delta(H) \geq 2$, the graph induced by the closed subwalk $v_2 \cdots v_\mu$ contains a circuit Q_3 of length at most $\mu - 2$. Furthermore, the circuit Q_3 is of length is at least 4, since the graph is triangle-free. Symmetrically, the graph induced by the closed subwalk $v_\mu \cdots v_r v_1 v_2$ contains a circuit Q_4 of length at most $r - \mu + 2$. So, it is obvious that $r \geq 8$. ■

Lemma 3.2. *Let H be a 3-connected graph embedded in a surface S. Assume that e is an edge of H incident with faces f' and f''. Then $f' = f''$ iff there is a non-contractible closed curve ϕ in the surface S such that $\phi \cap H$ is a single point. Furthermore, $f' = f''$ implies that $d_H(f') \geq 8$ if H is simple and triangle free.*

Proof. The first statement is obvious since H is 3-connected. By Lemma 3.1, the second statement holds as H is triangle free and $\delta(H) \geq 3$. ■

Definition 3.2. *Let H be a graph embedded in a surface S and C be a given dominating, chordless circuit of H. A good face f is normal (with respect to the circuit C) if f has only one vertex which is not on C. A good face f is special (with respect to the circuit C) if f is not normal.*

Examples of normal and special faces are illustrated in Fig. 1.

For the sake of convenience of later discussion, the following table lists Euler contributions of some edges where e is associated with $\{x, y, f', f''\}$:

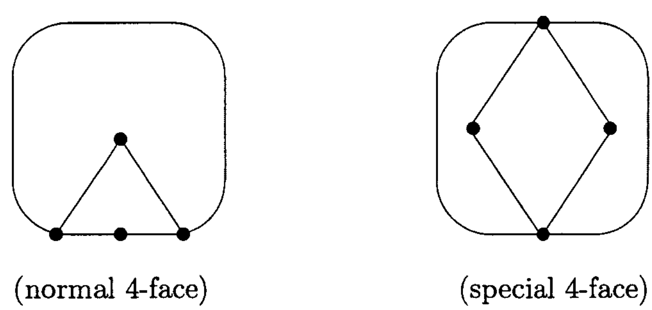

FIGURE 1. Example of a normal face and a special face.

$d_H(x)$	$d_H(y)$	$d_H(f')$	$d_H(f'')$	$\Phi(e)$
3	3	5	5	$\frac{1}{15}$
3	3	5	6	$\frac{1}{30}$
3	3	5	7	$\frac{1}{105}$
3	4	4	5	$\frac{1}{30}$
3	4	4	6	0
3	4	4	7	$-\frac{1}{42}$
3	4	5	5	$-\frac{1}{60}$
3	4	5	6	$-\frac{1}{20}$
3	4	5	7	$-\frac{31}{420}$
4	4	4	5	$-\frac{1}{20}$
4	4	4	6	$-\frac{1}{12}$
4	4	4	7	$-\frac{3}{28}$
4	4	5	6	$-\frac{2}{15}$

4. PROOF OF THE MAIN THEOREM

We prove the theorem by contradiction. Our proof consists of four parts. *Part* 1 gives some basic structures, *part* 2 discusses the existence of positive edges, *part* 3 describes five non-avoidable configurations, and *part* 4 implements charge–discharge on five non-avoidable configurations.

Let $C = v_1v_2 \cdots v_mv_1$ be a longest circuit of G without chord.

Let H be a graph obtained from G by contracting each C-bridge into a single vertex.

Part 1. Some basic structures.

(1) C is a longest circuit of H, and C is a dominating, chordless circuit of H.
(2) H is 3-connected. For each vertex $x \notin C$, $d_H(x) \geq 4$, since G is 4-connected. And, $d_H(x) \geq 3$ for each $x \in V(C)$.
(3) H is triangle free, for otherwise, C can be extended.
(4) C is of length at least 8, since $d_H(x) \geq 4$ for every $x \notin V(C)$.
(5) We investigate the local structure around some edges of C. Let $e = v_1v_2 \in E(C)$ and f' and f'' be the faces of H on the two sides of e (that is, e is associated with $\{v_1, v_2, f', f''\}$).

If

$$\max\{d_H(f'), d_H(f'')\} \leq 7 \text{ and } d_H(f') + d_H(f'') \leq 12,$$

then we claim that

(5-a) *f' and f'' are good faces and therefore, they are distinct.*

If, in addition, both f' and f'' are *normal* (see Definition 3.2), then we claim that ((5-b), (5-c), (5-d))

(5-b) $|E(f') \cap E(f'')| = 1$ (*that is, the edge $e = v_1v_2$ is the* only *common edge of those two faces. See Fig.* 2)

(5-c) $V(f') \setminus V(C) \neq V(f'')\setminus V(C)$.

(5-d) $d_H(f') + d_H(f'') \geq 10$.

***Proof of* (5-*a*).** Since f' and f'' are of degree ≤ 7, by Lemma 3.2, theirboundaries are circuits and therefore, they are good faces. Since they are good, the edge e cannot be passed twice by the boundary of any one of them, they must be different. ■

***Proof of* (5-*b*).** Let Q' (and Q'') be the maximal segment of C contained in the boundary of the normal face f' (and f'', respectively). Here, the edge $e \in Q' \cap Q''$.

Let P be a maximal segment of C contained in $Q' \cap Q''$. We first claim that *the length of P is at most one.* If not, then every internal vertex of P must be of degree 2, since f' and f'' are distinct and are on the two sides of the segment P. This contradicts that $\delta(H) \geq 3$ (by (2)).

Thus, we can see that the claim is true if $Q' \cap Q''$ contains only one segment. So, we assume that $Q' \cap Q''$ contains more than one segment.

Since Q', Q'' are contained in the circuit C, $Q' \cap Q''$ consists of at most two segments, say, P' and P''. Furthermore, $Q' \cup Q'' = C$. Note that we have already proved that each segment contained in $Q' \cap Q''$ is of length at most 1. So, if the claim is not true, each segment P' and P'' is of length precise 1. Hence,

$$\begin{aligned} |E(C)| &= |E(Q')| + |E(Q'')| - |E(P')| - |E(P'')| \\ &= (d_H(f') - 2) + (d_H(f'') - 2) - 2 \\ &\leq 12 - 6 = 6, \end{aligned}$$

This contradicts (4) that C is of length at least 8. ■

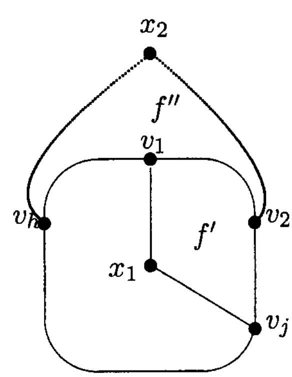

FIGURE 2. $Q' \cap Q''$ has one segment.

***Proof of* (5-*c*).** Let $x_1 \in V(f')\backslash V(C)$ and $x_2 \in V(f'')\backslash V(C)$. By (5-b), $e = v_1v_2$ is the only one edge contained in both f' and f''. Therefore, if $x_1 = x_2$, then $x_1v_1v_2x_1$ would be a triangle in the triangle-free graph H. A contradiction. ■

***Proof of* (5-*d*).** Let $f' = x_1v_1Cv_jx_1$ and $f'' = x_2v_2\bar{C}v_hx_2$, since $e = v_1v_2$ is the only edge in $E(f') \cap E(f'')$. (See Fig. 2.)

We have another circuit $C' = v_hx_2v_2v_1x_1v_jCv_h$. Here, with $h \geq j$, since C is a longest circuit in H, we have that

$$\begin{aligned}
|E(C)| &\geq |E(C')| \\
&= |E(C)| - |E(v_hCv_1)| - |E(v_2Cv_j)| + |E(v_hx_2v_2)| + |E(v_1x_1v_j)| \\
&= |E(C)| - (d_H(f') - 3) - (d_H(f'') - 3) + 4 \\
&= |E(C)| - d_H(f') - d_H(f'') + 10.
\end{aligned}$$

Hence,

$$d_H(f') + d_H(f'') \geq 10.$$

This proves our claim of (5-d). ■

(6) Let f be a face of degree 4. It is easy to prove that *if* $E(f) \cap E(C) \neq \emptyset$, *or some vertex of* f *is of degree* 3, *then* f *must be a normal* 4*-face.*

(7) We investigate the local structure of a degree 3 vertex of H. Let $v_i \in V(C)$ with $d_H(v_i) = 3$ and let $v_ix \in E(H)\backslash E(C)$. Let B be the C-bridge in the original graph G that the vertex x of H is created by the contraction of B, and let $v_j \in A(B)\backslash\{v_i\}$.

(7-a). We *claim* that *there is a path in B joining v_i and v_j of length at least* 3.

(7-b). We *claim* that $|E(v_iCv_j)| \geq 3$ *and* $|E(v_jCv_i)| \geq 3$. (See Fig. 3.)

(7-c). We *claim* that *each face of H containing the edge v_ix is of degree at least* 5.

(7-d). We *claim* that *if* $e = v_iv_{i+1} \in E(C)$ *with* $d_H(v_i) = d_H(v_{i+1}) = 3$, *then every face of H containing the edge e is of degree at least* 5.

***Proof of* (7-*a*).** Since $d_G(v_i) \geq 4$ and $d_H(v_i) = 3$, all vertices of $N_G(v_i)\backslash V(C)$ are in the bridge B, and, hence, $|I(B)| \geq 2$ in G. Let $I(B) = \{b_1, b_2, \ldots, b_t\}$ $(t \geq 2)$. Note that $d_B(v_i) \geq 2$ in B, since $d_G(v_i) \geq 4$. Without loss of generality, let $v_ib_1, v_ib_2 \in E(G)$. For $v_j \in A(B)\backslash\{v_i\}$, let $v_jb_k \in E(G)$ for some k $(1 \leq k \leq t)$, there are some paths joining v_i and v_j in B, since $B\backslash A(B)$ is connected. Let $P_{v_iv_j}$ be a longest path of B joining v_i and v_j in B. We *claim* that $|E(P_{v_iv_j})| \geq 3$. Otherwise, $|E(P_{v_iv_j})| \leq 2$ implies $|E(P_{v_iv_j})| = 2$. Let $P_{v_iv_j} = v_ib_kv_j$ and let

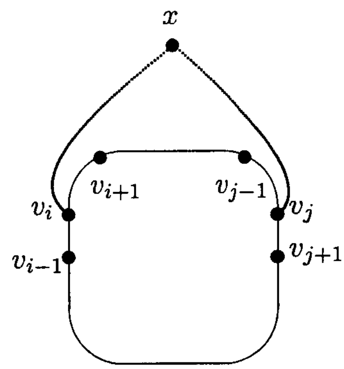

FIGURE 3. $E(v_i C v_j)| \geq 3$ and $|E(v_j C v_i)| \geq 3$.

$b_\ell \in \{b_1, b_2\} \backslash \{b_k\}$ as $d_B(v_i) \geq 2$. There is a path P' of $B \backslash A(B)$ joining b_ℓ and b_k, since $B \backslash A(B)$ is connected. The path $v_i b_\ell P' b_k v_j$ would be longer than $P_{v_i v_j}$, a contradiction. ■

***Proof of* (7-*b*).** It is obvious by (7-a) (see Fig. 3).

***Proof of* (7-*c*).** Assume that there is a face f of degree 4 where f is incident with the edge $v_i x$. We can see that f is neither a normal 4−face (by (7-b)), nor a special 4−face (by (6) since $d_H(v_i) = 3$).

***Proof of* (7-*d*).** Let $v_i x, v_{i+1} y \in E(H) \backslash E(C)$. By (7-c), each face incident with either $v_i x$ or $v_{i+1} y$ is of degree at least 5. Assume that there is a face f that contains the edge $e = v_i v_{i+1}$ but not any of $v_i x$ and $v_{i+1} y$. Since $d_H(v_i) = d_H(v_{i+1}) = 3$, the face f must use the segment $v_{i-1} v_i v_{i+1} v_{i+2}$ of the circuit C. So, the face f must be of length at least 5. ■

(8) Let edge $e = v_i v_{i+1} \in C$ be associated with $\{v_i, v_{i+1}, f', f''\}$ and $f' \neq f''$. We claim that

(8-a). *if both f' and f'' are normal faces, then*

$$\{d_H(f'),\, d_H(f'')\} \neq \{4,4\} \text{ or } \{4,5\};$$

(8-b). *if both f' and f'' are normal faces and $d_H(v_i) = d_H(v_{i+1}) = 3$, then*

$$\{d_H(f'), d_H(f'')\} \neq \{5,5\} \text{ or } \{5,6\}.$$

***Proof of* (8-*a*).** (8-a) is an immediate corollary of (5-d). ■

***Proof of* (8-*a*).** (Illustrations of (8-b) are in Fig. 4). Assume that $d_H\,(v_i) = d_H(v_{i+1}) = 3$ and $\{d_H(f'), d_H(f'')\} = \{5,5\}$. By (5-b), we know that $E(f') \cap E(f'') = \{e = v_i v_{i+1}\}$. Let $f' = w v_i v_{i+1}\ v_{i+2} v_{i+3} w$, $f'' = x v_{i-2} v_{i-1} v_i v_{i+1} x$ where

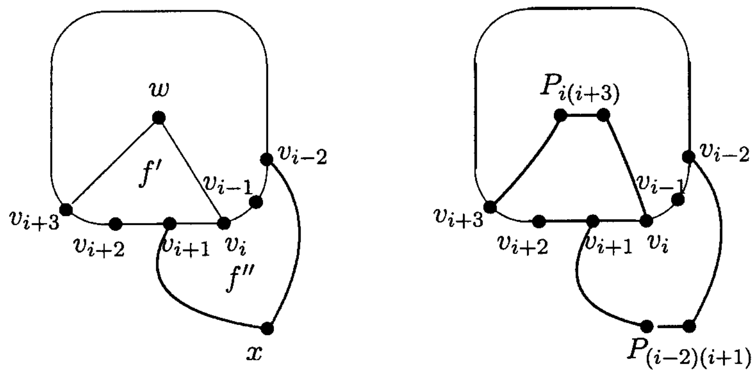

FIGURE 4. When $d_H(v_i) = d_H(v_{i+1}) = 3$ and $\{d_H(f'), d_H(f'')\} = \{5,5\}$.

$x, w \notin V(C)$ and $w \neq x$ (by (5-c)). By (7-a), there is a path $P_{(i-2)(i+1)}$ of length at least 3 in the C-bridge B_{i+1} of G corresponding to the vertex x of H, and there is a path $P_{i(i+3)}$ of length at least 3 in the C-bridge B_i of G corresponding to the vertex w of H. Then the circuit $v_{i-2}P_{(i-2)(i+1)}v_{i+1}v_iP_{i(i+3)}v_{i+3}\ Cv_{i-2}$ would be longer than C. This contradicts that C is a longest circuit of H.

Similarly, if $\{d_H(f'), d_H(f'')\} = \{5,6\}$, then C would not be a longest circuit of H, either.

Part 2. The existence of positive edges.

An edge e of H is *positive, negative, or zero* if $\Phi(e) > 0$, < 0, or $= 0$, respectively.

(9) We *claim* that *every non-negative edge e is incident with two distinct faces.*

Assume that e is incident with only one face f. Since H is triangle free and $\delta(H) \geq 3$, then $d_H(f) \geq 8$ by Lemma 3.1. We have that

$$\Phi(e) \leq \frac{1}{3} + \frac{1}{3} + \frac{1}{8} + \frac{1}{8} - 1 < 0.$$

(10) We *claim* that *there exists some positive edge in H.*

We prove it by contradiction. If the *claim* is false, then $\Phi(e) = 0$ for every edge $e \in E(H)$ since $\sum_{e \in E(H)} \Phi(e) = 0$.

There are only three possibilities:

$$\Phi(e) = \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} - 1 = 0, \text{ or} \tag{$*$}$$

$$\Phi(e) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - 1 = 0, \text{ or} \tag{$**$}$$

$$\Phi(e) = \frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} - 1 = 0, \tag{$***$}$$

since $d_H(x) \geq 3$ and H has no triangles.

Note that $(***)$ does not happen for any edge $e \in E(H) \backslash E(C)$, since $d_H(y) \geq 4$ for every $y \notin V(C)$.

(10.1) We *claim* that *every vertex of* $V(C)$ *is of degree* 4.

Assume that $d_H(x) = 3$ for some vertex x of H. By (2), $x \in V(C)$.

Let $e = xy \in E(H) \backslash E(C)$ $(x \in V(C), y \notin V(C))$, let f' and f'' be the faces on the two sides of edge $e = xy$. Since $d_H(x) = 3$ then we must have the case $(*)$ here:

$$\Phi(e) = \frac{1}{d_H(x)} + \frac{1}{d_H(y)} + \frac{1}{d_H(f')} + \frac{1}{d_H(f'')} - 1 = \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} - 1 = 0 \qquad (*)$$

as $d_H(x) = 3$ and $d_H(y) \geq 4$.

(10.1.1) We *claim* that *the case* $(*)$ *will not happen.* Assume that the case $(*)$ holds for an edge $e \in E(H) \backslash E(C)$, then there is at least one 4-face incident with e. This contradicts (7-c), since $d_H(x) = 3$ and the face f' contains the edge in *calculation* : xy.

(10.1.2) By (10.1.1), only the case $(**)$ holds for every $e = xy \in E(H) \backslash E(C)$.

The case $(**)$ implies $d_H(x) = 4$ for all $x \in V(C)$, since each vertex of $V(C)$ is incident with at least one edge of $E(H) \backslash E(C)$. So, by (10.1.1) and (10.1.2), $d_H(x) = 4$ for every $x \in V(C)$.

(10.2) Now we consider edges in $E(C)$. Let $e(= v_1 v_2) \in C$ be associated with $\{v_1, v_2, f', f''\}$, by (10.1), $d_H(v_1) = d_H(v_2) = 4$, we must have

$$\Phi(e) = \frac{1}{d_H(v_1)} + \frac{1}{d_H(v_2)} + \frac{1}{d_H(f')} + \frac{1}{d_H(f'')} - 1 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - 1 = 0$$

as H is triangle free. So, $d_H(f') = d_H(f'') = 4$. Obviously, both 4-faces f' and f'' must be normal faces by (6). But, this contradicts (8-a). Hence, there exists at least one edge with non-zero Euler contribution. By the equation (4) of Section 2, there are some positive edges.

(11) We *claim* that *every positive edge must be on the longest circuit* C.

By (10), there is some positive edge. Assume that our claim is false. Let $e = wv \in E(H) \backslash E(C)$ be a positive edge associated with $\{v, w, f', f''\}$ and $v \in V(C), w \notin V(C)$.

Notice that $d_H(v) \geq 3$, $d_H(w) \geq 4$ by (2).

(11.1). We *claim* that $d_H(v) = 3$. If not, assume $d_H(v) \geq 4$, then

$$\Phi(e) = \frac{1}{d_H(v)} + \frac{1}{d_H(w)} + \frac{1}{d_H(f')} + \frac{1}{d_H(f'')} - 1 \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - 1 \leq 0$$

as $d_H(w) \geq 4$ and H is triangle-free, a contradiction.

(11.2). We *claim* that $d_H(f'), d_H(f'') \geq 5$.

By (11.1), we have that $d_H(v) = 3$. Therefore, the claim follows immediately by (7-c).

By (11.1) and (11.2), we have that

$$\Phi(e) \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{5} - 1 < 0,$$

This contradicts that e is a positive edge.

Part 3. Five non-avoidable configurations.

(12) We investigate the local structure of the face(s) incident with a positive edge e.

By (11), each positive edge e must be on the longest circuit C. Hence, every edge not on C is a non-positive edge. Without loss of generality, let $e \in E(C)$ be a positive edge associated with $\{v_1, v_2, f', f''\}$.

If both $d_H(v_1), d_H(v_2) \geq 4$ or one of them is of at least 6, then

$$\Phi(e) = \frac{1}{d_H(v_1)} + \frac{1}{d_H(v_2)} + \frac{1}{d_H(f')} + \frac{1}{d_H(f'')} - 1 \leq 0,$$

since both $d_H(f'), d_H(f'') \geq 4$. So,

$$\min\{d_H(v_1), d_H(v_2)\} = 3 \text{ and } \max\{d_H(v_1), d_H(v_2)\} \leq 5$$

Thus,

$$\{d_H(v_1), d_H(v_2)\} \in \{\{3,3\}, \{3,4\}, \{3,5\}\}$$

where we assume that $d_H(v_1) \leq d_H(v_2)$.

(12.1) We first investigate the case $d_H(v_1) = 3$ and $d_H(v_2) \geq 4$. That is,

$$\{d_H(v_1), d_H(v_2)\} = \{3,4\} \text{ or } \{3,5\}.$$

(a)

$$\min\{d_H(f'), d_H(f'')\} \geq 4$$

since H is triangle-free.

$$\max\{d_H(f'), d_H(f'')\} \leq 5,$$

for otherwise, $\Phi(e) \leq 0$. If $\{d_H(f'), d_H(f'')\} = \{5,5\}$, then $\Phi(e) < 0$. So,

$$\{d_H(f'), d_H(f'')\} = \{4,4\} \text{ or } \{5,4\}.$$

(b) By (a), one of f', f'' is a 4-face. Any 4-face must be normal (by (6)). And note that at least one of f', f'' is special (by (8-a)). Hence, only one case left by (a) and (b):

$\{d_H(f'), d_H(f'')\} = \{5, 4\}$ *where* f' *is a special* 5*–face*, f'' *is a normal* 4*–face and* $\{d_H(v_1), d_H(v_2)\} = \{3, 4\}$ *or* $\{3, 5\}$
(*call it* "Configuration 1," *see Fig.* 5).

(12.2) We, then, investigate the case that $d_H(v_1) = d_H(v_2) = 3$.

By (7-d), none of $\{f', f''\}$ is a 4-face. So, $d_H(f'), d_H(f'') \geq 5$.

If both $d_H(f'), d_H(f'')$ are at least 6, or one of $\{d_H(f'), d_H(f'')\}$ is at least 8, then $\Phi(e) \leq 0$. So, we have

$$\min\{d_H(f'), d_H(f'')\} = 5 \text{ and } \max\{d_H(f'), d_H(f'')\} \leq 7.$$

Therefore, there are only three possible subcases left in this case:

$$\{d_H(f'), d_H(f'')\} \in \{\{5, 5\}, \{5, 6\}, \{5, 7\}\}.$$

(a) By (8-b), we have that f' *and* f'' *cannot be both normal faces if* $\{d_H(f'), d_H(f'')\} = \{5, 5\}$ *or* $\{5, 6\}$.

(b) We claim that f' *and* f'' *cannot be both special if* $\{d_H(f'), d_H(f'')\} = \{5, 5\}$ *or* $\{5, 6\}$. Otherwise, assume both f' and f'' are special. Since f' is a special 5-face, the boundary of f' must be $v_1v_2xv_iyv_1$ for some $x, y \notin V(C)$ and $v_i \in V(C)$. Thus, the boundary of the face f'' must use the segment $v_mv_1v_2v_3$ of C, since $d_H(v_1) = d_H(v_2) = 3$. This would imply that $d_H(f'') \geq 7$ if f'' is special.

From the discussion above, we know that, except for the case that

$$\{d_H(f'), d_H(f'')\} = \{5, 7\},$$

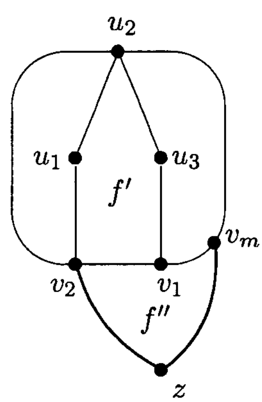

FIGURE 5. Configuration 1.

the positive edge e is incident with precisely one special face. Let f' be the special face if only one of $\{f', f''\}$ is special, or f' is the shorter one if both f' and f'' are special, f' is the longer one if both f' and f'' are normal.

Above discussion lead us to the following "Configurations":

$$d_H(f') = 5 \textit{ and } d_H(f'') \in \{6, 7\} \ (\textit{where } f' \textit{ is special}) \textit{ and } d_H(v_1) \\ = d_H(v_2) = 3 \ (\textit{call it } \text{"Configuration 2," } \textit{see Fig. } 6).$$

$$d_H(f') \in \{6, 7\} \textit{ and } d_H(f'') = 5 \ (\textit{where } f' \textit{ is special and } f'' \textit{ is normal}) \\ \textit{and } d_H(v_1) = d_H(v_2) = 3 \ (\textit{call it } \text{"Configuration 3," } \textit{see Fig. } 7).$$

$$d_H(f') = d_H(f'') = 5 \ (\textit{where } f' \textit{ is special, and } f'' \textit{ is normal}) \textit{ and} \\ d_H(v_1) = d_H(v_2) = 3 \ (\textit{call it } \text{"Configuration 4," } \textit{see Fig. } 8).$$

$$d_H(f') = 7 \textit{ and } d_H(f'') = 5 \textit{ and both } f' \textit{ and } f'' \textit{ are normal and} \\ d_H(v_1) = d_H(v_2) = 3 \ (\textit{call it } \text{"Configuration 5," } \textit{see Fig. } 9).$$

Part 4. Charge–discharge

(13) From the previous subsections, we have found that the graph H embedded in the torus or the Klein bottle must have some positive edges and they are contained in the longest circuit C.

Let $\psi : E(H) \to R$ be a function such that $\psi(e) = \Phi(e)$ (Φ is the Euler contribution) as the initial charge to $E(H)$. As we already knew in Section 2,

$$\sum_{e \in E(H)} \psi(e) = 0 \qquad \text{(A)}$$

In this subsection, we will redistribute (charge and discharge, as commonly called) the function ψ such that the total sum of the function $\psi(e)$ remains the

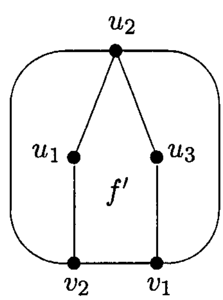

FIGURE 6. Configuration 2.

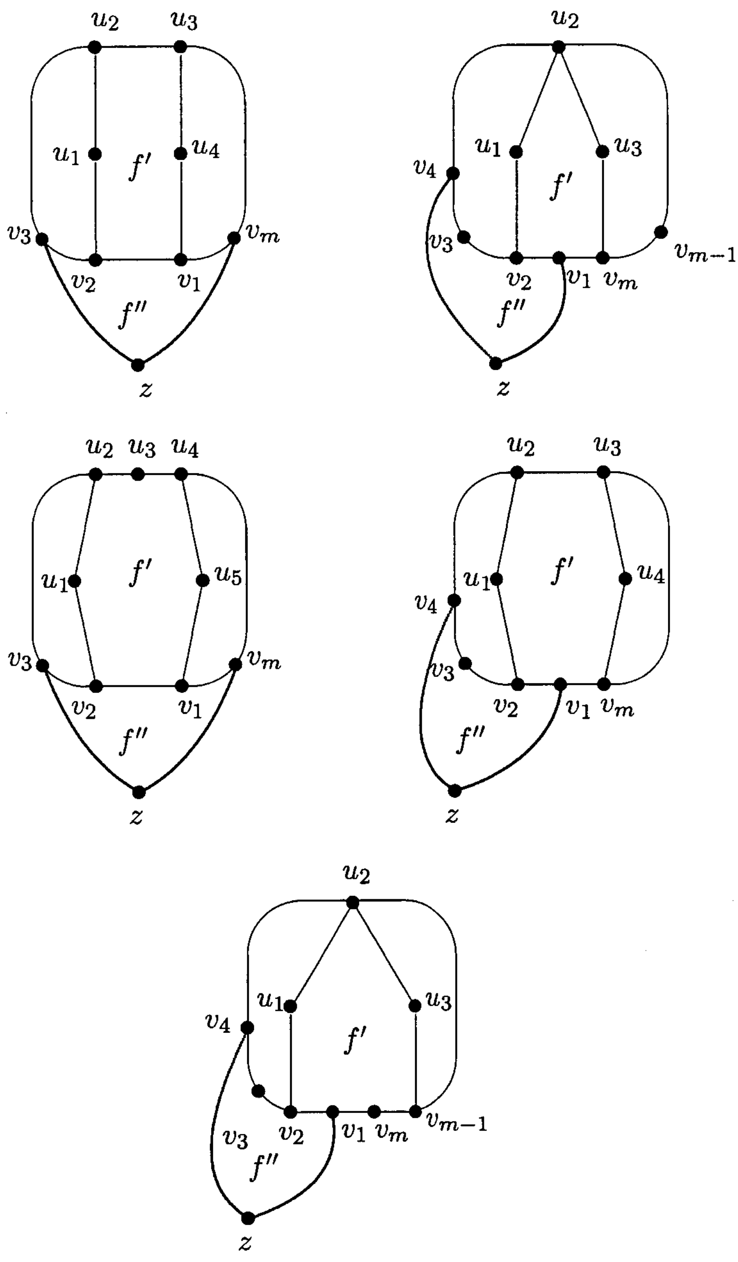

FIGURE 7. Some examples of Configuration 3.

same as before, and we will show later that under the new function, the total sum of the function will be negative. This will contradict to Equation (A).

For the sake of convenience of later discussion, we define some terms. Under the function $\psi = \Phi$, an edge e with $\psi(e) > 0$ is called a $\mathcal{D}$-edge (means that this edge will be discharged later). We notice that in Configurations 1–4, a $\mathcal{D}$-edge is

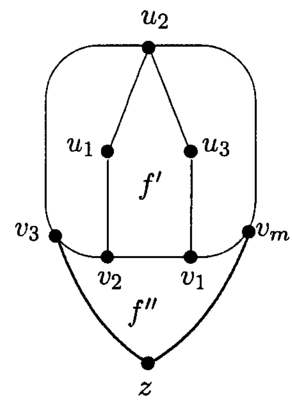

FIGURE 8. Configuration 4.

incident with a special face f', which is called a $\mathcal{C}/\mathcal{D}$-face (it means that the charge/discharge operation will occur along the edges of this face). We also notice that in Configuration 5, the faces f', f'' incident with the $\mathcal{D}$-edge e are normal, which, are called $\mathcal{C}/\mathcal{D}$-faces, in this case.

Each $e \notin E(C)$ is called a $\mathcal{C}$-edge, $\mathcal{C}$-edge e is called a $[++]$-edge if it is incident with two $\mathcal{C}/\mathcal{D}-$faces, or is a $[+-]$-edge if it is incident with at most one $\mathcal{C}/\mathcal{D}$-face (it means that the edge e will be charged twice from two sides or charged at most once from one side).

The charge/discharge operation is described as follows.

For a $\mathcal{C}/\mathcal{D}$-face f' in configurations 1–4, we need to show that

$$\sum_{e \in E(f')} \varepsilon(e)\psi(e) < 0$$

where $\varepsilon(e) = 1$ if e is a $\mathcal{D}$-edge or e is a $[+-]$-edge, and $\varepsilon(e) = 1/2$ if e is a $[++]$-edge. (That is, each edge $e \in E(f) \backslash \{\mathcal{D}\text{-}edges\}$, which is of negative value originally, will be charged with $\varepsilon(e)|\psi(e)|$ from those $\mathcal{D}$-edges along the face f'.

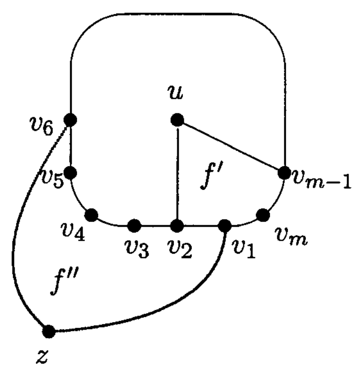

FIGURE 9. Configuration 5.

And we will show that those $[++]$ and $[+-]$-edges remain non-positive, and those $\mathcal{D}$-edges will be of negative value after the operation.) For Configuration 5, let $e = v_1v_2$ be the $\mathcal{D}$-edge and let v_2u and v_1z be the edges of $E(H)\backslash E(C)$ that are incident with $e = v_1v_2$. The charge/discharge operation occurs only among the edges $\in \{v_1v_2, v_2u, v_1z\}$ as follows: each $e^* \in \{v_1z, v_2u\}$ will be charged $\frac{1}{2}|\psi(e^*)|$ from the $\mathcal{D}$-edge v_1v_2.

Note that Configuration 5 is very different that no face is special, it will be dealt with separately. And Configuration 4 needs a little more attention, since a rough estimation would not lead us to a negative total value of ψ around the special face. In the subsections (13-1) and (13-2), we will deal with the first four configurations.

(13-1) *The calculation of*

$$\sum_{e \in E(f') \cap E(C)} \psi(e)$$

for Configurations 1–4.

Since f' is a special face of degree at most 7 and contains at least one edge $e = v_1v_2$ of the chordless circuit C,

$$|E(f')\backslash E(C)| = 2|V(f') - V(C)|$$

must be even and, therefore, is either 4 or 6 (the later case occurs only when $d_H(f') = 7$). Since no edge of $E(H)\backslash E(C)$ is positive (by (11)), they are $\mathcal{C}$-edges. Considering the worst case in calculations, each edge of $E(f') \cap E(C)$ could be positive (therefore, a $\mathcal{D}$-edge).

(i) If the degree of the special face f' is 5, then $e = v_1v_2$ is the only edge of $E(f') \cap E(C)$. Hence,

(i-1) for Configuration 1,

$$\sum_{e \in E(f') \cap E(C)} \psi(e) = \psi(v_1v_2) \leq 1/3 + 1/4 + 1/5 + 1/4 - 1 = 1/30;$$

(i-2) for Configuration 2,

$$\sum_{e \in E(f') \cap E(C)} \psi(e) = \psi(v_1v_2) \leq 1/3 + 1/3 + 1/5 + 1/6 - 1 = 1/30;$$

(i-3) for Configuration 4,

$$\sum_{e \in E(f') \cap E(C)} \psi(e) = \psi(v_1v_2) \leq 1/3 + 1/3 + 1/5 + 1/5 - 1 = 1/15.$$

(ii) For Configuration 3, let $e = v'v'' \in E(f') \cap E(C)$ be associated with $\{v', v'', f', f^*\}$. By applying (7-c) and (7-d) whenever there is a possibility that a degree 3 vertex or a pair of degree 3 vertices is involved, the sequence of degrees $\{d_H(v'), d_H(v''), d_H(f'), d_H(f^*)\}$ must be one of following cases:

$$\{3, 3, d_H(f'), \geq 5\}, \{3, \geq 4, d_H(f'), \geq 4\}, \{\geq 4, \geq 4, d_H(f'), \geq 4\}.$$

Hence,

$$\psi(e) \leq \max\left\{\frac{1}{3}+\frac{1}{3}+\frac{1}{d_H(f')}+\frac{1}{5}-1, \frac{1}{3}+\frac{1}{4}+\frac{1}{d_H(f')} +\frac{1}{4}-1, \frac{1}{4}+\frac{1}{4}+\frac{1}{d_H(f')}+\frac{1}{4}-1\right\} = \frac{1}{3}+\frac{1}{3}+\frac{1}{d_H(f')}+\frac{1}{5}-1 = \frac{1}{d_H(f')}-\frac{2}{15}.$$

So, if $|E(f') \cap E(C)| = d_H(f') - 4$, then

$$\begin{aligned}\sum_{e \in E(f') \cap E(C)} \psi(e) &\leq [d_H(f') - 4] \cdot [1/d_H(f') - 2/15] \\ &\leq [6-4] \cdot [1/6 - 2/15] = 1/15\end{aligned}$$

(since $d_H(f') = 6$ or 7); if $|E(f') \cap E(C)| = d_H(f') - 6$ and $d_H(f') = 7$, then

$$\begin{aligned}\sum_{e \in E(f') \cap E(C)} \psi(e) &\leq [d_H(f') - 6] \cdot [1/d_H(f') - 2/15] \\ &< 1/15.\end{aligned}$$

(13-2) *The calculation of*

$$\sum_{e \in E(f') \setminus E(C)} \psi(e)$$

for Configurations 1–4.

We notice that the special face f' has exactly four or six edges in $E(f') \setminus E(C)$ (the later case occurs only for Configuration 3 when $d_H(f') = 7$).

(i) For Configuration 1, the degree of f' is 5. Let $f' = v_1 v_2 u_1 u_2 u_3 v_1$, where $u_1, u_3 \notin V(C)$. Here, $d_H(v_1) = 3$ and $d_H(v_2) \geq 4$. Then the degree sequence $\{d_H(v_1), d_H(u_3), d_H(f'), d_H(f^*)\}$ associated with the edge $v_1 u_3$ must be $\{3, \geq 4, 5, \geq 5\}$ by (2) and (7-c); and the degree sequence $\{d_H(x), d_H(y), d_H(f'), d_H(f^*)\}$ associated with the edge $xy \in \{v_2 u_1, u_1 u_2, u_2 u_3\}$ must be $\{\geq 4, \geq 4, 5, \geq 4\}$ by (2) and (3). Thus,

$$\psi(v_1 u_3) \leq 1/3 + 1/4 + 1/5 + 1/5 - 1 = -1/60,$$

and

$$\psi(v_2u_1), \psi(u_2u_1), \psi(u_2u_3) \leq 1/4 + 1/4 + 1/5 + 1/4 - 1 = -1/20.$$

So,

$$\sum_{e \in E(f')\setminus E(C)} \psi(e) \leq -1/60 + 3 \cdot (-1/20) = -1/6.$$

(ii) For Configurations 2 and 4, the degree of f' is 5. Let $f' = v_1v_2u_1 u_2u_3v_1$ where $u_1, u_3 \notin V(C)$. Then the degree sequence $\{d_H(x),\ d_H(y),\ d_H(f'),\ d_H(f^*)\}$ associated with the edge $xy \in \{v_2u_1, v_1u_3\}$ must be $\{3, \geq 4, 5, \geq 5\}$ by (2) and (7-c); and the degree sequence $\{d_H(x), d_H(y),\ d_H(f'),\ d_H(f^*)\}$ associated with the edge $xy \in \{u_1u_2, u_2u_3\}$ must be $\{\geq 4, \geq 4, 5, \geq 4\}$ by (2) and (3). Thus,

$$\psi(v_2u_1), \psi(v_1u_3) \leq 1/3 + 1/4 + 1/5 + 1/5 - 1 = -1/60,$$

and

$$\psi(u_1u_2), \psi(u_2u_3) \leq 1/4 + 1/4 + 1/5 + 1/4 - 1 = -1/20.$$

So, the total ψ value of those $\mathcal{C}$-edges is

$$\sum_{e \in E(f')\setminus E(C)} \psi(e) \leq 2 \cdot (-1/60) + 2 \cdot (-1/20) = -2/15.$$

(iii) For Configuration 3, the degree of f' is 6 or 7. Let $e = v'v'' \in E(f')\setminus E(C)$ be associated with $\{v', v'', f', f^*\}$. By applying (7-c), whenever there is a possibility that a degree 3 vertex is involved, the sequence of degrees $\{d_H(v'), d_H(v''), d_H(f'), d_H(f^*)\}$ must be one of following cases:

$$\{3, \geq 4, d_H(f'), \geq 5\}, \{\geq 4, \geq 4, d_H(f'), \geq 4\}.$$

Hence,

$$\begin{aligned}\psi(e) \leq \max\{&1/3 + 1/4 + 1/d_H(f') + 1/5 - 1, 1/4 + 1/4 \\ &+ 1/d_H(f') + 1/4 - 1\} = 1/3 + 1/4 + 1/d_H(f') \\ &+ 1/5 - 1 \leq 1/d_H(f') - 13/60 \leq 1/6 - 13/60 = -1/20.\end{aligned}$$

Since $|E(f')\setminus E(C)| = 4$ or 6,

$$\sum_{e \in E(f')\setminus E(C)} \psi(e) \leq 4[-1/20] = -1/5.$$

(13-3) By the calculations in (13-1) and (13-2), we are ready to estimate $\sum_{e\in E(f')}\varepsilon(e)\psi(e)$ for some configurations.

Note that by considering the worst case in the calculation, each $\mathcal{C}$-edge should be considered as a $[++]$-edge. That is, the coefficient ε will be $1/2$ for the worst cases in the estimation.

(i) For Configuration 1 (by (13-1)-(i-1) and (13-2)-(i)), we have that

$$\sum_{e\in E(f')}\varepsilon(e)\psi(e)\le\sum_{e\in E(f')\cap E(C)}\varepsilon(e)\psi(e)+\sum_{e\in E(f')\setminus E(C)}\varepsilon(e)\psi(e)$$
$$\le(1/30)+(1/2)(-1/6)=-1/20.$$

(ii) For Configuration 2 (by (13-1)-(i-2) and (13-2)-(ii)), we have that

$$\sum_{e\in E(f')}\varepsilon(e)\psi(e)\le\sum_{e\in E(f')\cap E(C)}\varepsilon(e)\psi(e)+\sum_{e\in E(f')\setminus E(C)}\varepsilon(e)\psi(e)$$
$$\le(1/30)+(1/2)(-2/15)=-1/30.$$

(iii) For Configuration 3, (by (13-1)-(ii) and (13-2)-(iii)), we have that

$$\sum_{e\in E(f')}\varepsilon(e)\psi(e)\le\sum_{e\in E(f')\cap E(C)}\varepsilon(e)\psi(e)+\sum_{e\in E(f')\setminus E(C)}\varepsilon(e)\psi(e)$$
$$\le(1/15)+(1/2)(-1/5)=-1/30.$$

The value of $\sum_{e\in E(f')}\varepsilon(e)\psi(e)$ for each Configuration 1, 2, and 3 is negative. However, the same estimation for Configuration 4 would give us a *zero*. This is not what we would like to have. Of course, we notice that the estimations in (13-1) and (13-2) are not very tight at all. Therefore, some further attention is needed for Configuration 4 (see Fig. 8).

(13-4) For Configuration 4, the degree of both faces $f'=v_1v_2u_1u_2u_3v_1$ (special) and $f''=v_1v_2v_3zv_mv_1$ (normal) are 5 and $d_H(v_1)=d_H(v_2)=3$, where $u_1,u_3\notin V(C)$. Here, $d_H(v_1)=d_H(v_2)=3$.

Let f' and $f_{u_3v_1}$ be the faces on the two sides of edge u_3v_1. By (7-c), $d_H(f_{u_3v_1})\ge 5$ since $d_H(v_1)=3$.

Case 4-1. If $d_H(f_{u_3v_1})\ge 6$.

For the $\mathcal{D}$-edge $e=v_1v_2$,

$$\psi(v_1v_2)=\frac{1}{3}+\frac{1}{3}+\frac{1}{5}+\frac{1}{5}-1=\frac{1}{15}.$$

Note that $d_H(f_{u_3v_1})\ge 6$,

$$\psi(u_3v_1)\le\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-1=-\frac{1}{20}.$$

By earlier results (see (13-2)-(ii)), we have that

$$\psi(u_1u_2), \psi(u_2u_3) \leq -\frac{1}{20}$$

and

$$\psi(v_2u_1) \leq -\frac{1}{60}.$$

So, for all the $\mathcal{C}$-edges, we have that

$$\sum_{e\in E(f')\setminus E(C)} \psi(e) \leq -\left[\frac{1}{60}+\frac{1}{20}+\frac{1}{20}+\frac{1}{20}\right] = -\frac{10}{60}.$$

Hence,

$$\sum_{e\in E(f')} \varepsilon(e)\psi(e) \leq \frac{1}{2}(-\frac{10}{60}) + \frac{1}{15} < 0.$$

Case 4-2. If $d_H(f_{u_3v_1}) = 5$.

(a) We shall determine that the $\mathcal{C}$-edge u_3v_1 must be a $[+-]$-edge (thus, the coefficient ε would be 1 instead of 1/2).

Assume that u_3v_1 is a $[++]$-edge, hence $f_{u_3v_1}$ is a $\mathcal{C}/\mathcal{D}$-face with respect to some edges of $E(f_{u_3v_1}) \cap E(C)$.

First, we show that $f_{u_3v_1}$ *cannot be normal.* Assume that the face $f_{u_3v_1}$ is normal. Thus, the normal 5-face $f_{u_3v_1}$ has the boundary $v_1u_3v_{m-2}v_{m-1}v_mv_1$. By (7-a), there is a path P of G joining v_{m-2} and v_1 of length at least 3, where $E(P) \cap E(C) = \emptyset$. Then $v_{m-2}Pv_1v_mzv_3Cv_{m-2}$ (see the figure of Configuration 4) would be longer than C, a contradiction. So, the face $f_{u_3v_1}$ must be special.

Since $f_{u_3v_1}$ is special and $d_H(v_1) = 3$ and $d_H(f_{u_3v_1}) = 5$ (for this case 4-2), v_mv_1 is the only edge in $E(f_{u_3v_1}) \cap E(C)$. Furthermore, $d_H(v_m) \geq 4$.

Now, with a calculation, we have that $\psi(v_mv_1) < 0$. Since v_mv_1 is the only edge in $E(f_{u_3v_1}) \cap E(C)$, the face $f_{u_3v_1}$ cannot be a $\mathcal{C}/\mathcal{D}$-face and therefore, it proves our claim that u_3v_1 is a $[+-]$-edge.

(b) *Calculations:* For the $\mathcal{D}$-edge v_1v_2,

$$\psi(v_1v_2) \leq \frac{1}{3}+\frac{1}{3}+\frac{1}{5}+\frac{1}{5}-1 = \frac{1}{15}.$$

Now, take a half charge of all other $\mathcal{C}$-edges and total charge of the $[+-]$-edge u_3v_1, we have that

$$\sum_{e\in E(f')} \varepsilon(e)\psi(e) = \psi(v_1v_2) + \frac{1}{2}[\psi(v_2u_1) + \psi(u_1u_2) + \psi(u_2u_3)] + \psi(u_3v_1)$$

$$= \frac{1}{15} - \frac{1}{2}\left[\frac{1}{60}+\frac{1}{20}+\frac{1}{20}\right] - \frac{1}{60} = \frac{1}{15} - \frac{9}{120} < 0.$$

(14) **Configuration 5** (Fig. 9). $\{d_H(f'), d_H(f'')\} = \{5, 7\}$ where f' is a normal 5-face and f'' is a normal 7-face.

By (5-b), $E(f') \cap E(f'') = e(= v_1 v_2)$, without loss of generality, let $f' = v_1 v_2 u v_{m-1} v_m v_1$ (where $u \notin V(C)$) and $f'' = z v_1 v_2 \cdots v_6 z$ (where $z \notin V(C)$).

Note that the face f' (or, f'' as well) is not a $\mathcal{C}/\mathcal{D}$-face for any edge of $E(f') \cap E(C)$ with respect to any Configuration 1–4, since the $\mathcal{C}/\mathcal{D}$—faces for Configurations 1–4 must be special. Thus, if the charge/discharge operation occurs more than once in the face f' (or, similar for f''), it must be for the $\mathcal{D}$-edge $v_{m-1} v_m$ or $(v_5 v_6)$ in another Configuration 5. There is no conflict in the calculation here, since the charge/discharge operation occurs only at incident $\mathcal{C}$-edges for Configuration 5. Let the edge $v_1 z$ be associate with $\{v_1, z, f'', f_{v_1 z}\}$ and the edge $v_2 u$ be associate with $\{v_2, u, f', f_{v_2 u}\}$. By (7-c), $d_H(f_{v_1 z}) \geq 5, d_H(f_{v_2 u}) \geq 5$, since $d_H(v_1) = d_H(v_2) = 3$.

So, we have following calculation:

$$\begin{aligned}
\psi(v_1 v_2) &= \frac{1}{3} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - 1 = \frac{1}{105}. \\
\psi(v_2 u) &\leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{5} - 1 = -\frac{1}{60}, \\
\psi(v_1 z) &\leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{7} - 1 = -\frac{31}{420}, \\
\psi(v_1 z) + \psi(v_2 u) &\leq -\frac{38}{420}.
\end{aligned}$$

Therefore,

$$\sum_{e \in E(f')} \varepsilon(e)\psi(e) \leq \psi(v_1 v_2) + \frac{1}{2}\psi(v_1 z) + \frac{1}{2}\psi(v_2 u) < 0.$$

Final conclusion. After charging/discharging, we have seen that the total charge of all the edges is negative. This contradicts to the fact

$$\sum_{e \in E(H)} \psi(e) = 0$$

(Equation (4) of Section 2).

5. REMARKS

The proof of the main theorem actually shows that every longest circuit of a 4-connected graph embedded in a surface with a non-negative characteristic has a chord. However, it is already known that every 4-connected graph embedded in a sphere or projective plane (Tutte [5] and Thomas and Yu [6]) is hamiltonian.

But it remains open (Grünbaum [2] and Nash-Williams [4]) that every 4-connected toroidal graph is hamiltonian and it was proved for 5-connected toroidal graphs (Thomas and Yu [7]).

REFERENCES

[1] B. Alspach and C. Godsil, Cycles in graphs, Ann Discrete Math 27 (1985), 461–467.

[2] B. Grünbaum, Polytopes, graphs, and complexes, Bull Amer Math Soc 76 (1970), 1131–1201.

[3] H. Lebesgue, Quelques conséquences simples de la formule d'Euler, J de Math, 9, Sér 19 (1940), 27–43.

[4] C. St. J. A. Nash-Williams, Unexplored and semi-explored territories in graph theory, New directions in graph theory, Academic Press, New York, 1973, pp. 169–176

[5] W.T. Tuttle, A theorem on planar graphs, Trans Amer Math Soc 82 (1956), 99–116.

[6] R. Thomas and X. Yu, 4-connected projective-planar graphs are hamiltonian, J Combin Theory Ser B 62 (1994), 114–132.

[7] R. Thomas and X. Yu, Five-connected toroidal graphs are hamiltonian, J Combin Theory Ser B 69 (1997), 79–96.

[8] C. Thomassen, Chords of longest cycle in cubic graphs, J Combin Theory Ser B 71 (1997), 215–230.

[9] C.-Q. Zhang, Longest cycles and their chords, J Graph Theory 11 (1987), 521–529.