Circular Flows of Nearly Eulerian Graphs and Vertex-Splitting

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Abstract: The odd edge connectivity of a graph G, denoted by $\lambda_o(G)$, is the size of a smallest odd edge cut of the graph. Let S be any given surface and ϵ be a positive real number. We proved that there is a function $f_S(\epsilon)$ (depends on the surface S and $\lim_{\epsilon\to 0} f_S(\epsilon) = \infty$) such that any graph G embedded in S with the odd-edge connectivity at least $f_S(\epsilon)$ admits a nowhere-zero circular $(2+\epsilon)$-flow. Another major result of the work is a new vertex splitting lemma which maintains the old edge connectivity and graph embedding. J Graph Theory 40: 147–161, 2002

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1. INTRODUCTION

A. Real Line Extension of Integer Flow

The concept of integer flow was introduced by Tutte [15, 16] and is the dual of the vertex coloring notion for planar graphs.

Definition 1.1. *Let $G = (V, E)$ be a graph. An ordered pair (D, f) is called an integer flow of G if D is an orientation of $E(G)$ and $f : E(G) \mapsto Z$, the set of integers, such that the total in-flow equals the total out-flow at every vertex. An integer flow (D, f) is a k-flow if $|f(e)| \leq k - 1$ for every edge e of G. And an integer flow is nowhere-zero if $f(e) \neq 0$ for every edge e of G.*

It was proved by Tutte ([16]) that a planar graph G is face-k-colorable if and only if G admits a nowhere-zero integer k-flow.

Similar to the circular chromatic number ([17], or see [2], [20]) which extends the measuring of the conventional chromatic number, the concept of the integer flow was extended to rational values ([3], or see [19]) as follows.

Definition 1.2 [3]. [1]*Let $\frac{k}{d}$ be a rational number ($\frac{k}{d} \geq 2$) with k and d as integers. A nowhere-zero circular $\frac{k}{d}$-flow (D, f) of a graph G is an integer flow such that $d \leq |f(e)| \leq k - d$.*

The following properties of circular $\frac{k}{d}$-flows indicate its relation with integer flows.

(1) For an integer $r = \frac{k}{d}$, a graph G admits a nowhere-zero integer r-flow if and only if G admits a nowhere-zero circular $\frac{k}{d}$-flow;
(2) For any pair of real numbers $\frac{k_1}{d_1} \geq \frac{k_2}{d_2} \geq 2$, a graph G admits a nowhere-zero $\frac{k_1}{d_1}$-flow if G admits a nowhere-zero $\frac{k_2}{d_2}$-flow (where either flow can be circular or integer).

Similar to the circular chromatic number, we define the *flow index* as follows.

Definition 1.3. *Let G be a graph. The flow index of G, denoted by $\phi(G)$, is the least real number r that G admits nowhere-zero r-flow (either circular or integer).*

Since all graphs we consider here are finite, it is easy to show that r exists for every bridgeless graph and is a rational number at least 2.

[1]The circular flow defined here was called *fractional flow* in [3] and [19]. However, it was pointed out by some colleagues that the adjective "*fractional*" is commonly reserved for linear programming relaxations of integer programming problems. Therefore, it would not be appropriate to called it "fractional flow," since it is actually a rational number extension instead of linear programming relaxation. Here, the adjective "*circular*" is adopted from its dual problem—"*circular coloring*."

B. Odd-Edge Connectivity

Definition 1.4. *The odd-edge connectivity of a graph G, denoted by $\lambda_o(G)$, is the size of a smallest odd edge cut of G.*

It was conjectured by Tutte [1,14,16], or see [19]) that every graph with odd-edge connectivity at least 3 admits a nowhere-zero 5-flow and every graph with odd-edge connectivity at least 5 admits a nowhere-zero 3-flow. And it is a well-known fact that every eulerian graph (whose odd-edge connectivity is ∞) admits a nowhere-zero 2-flow. One may feel that the odd edge cuts of a graph play a central role in the study of the flow problem:

the greater the odd-edge connectivity λ_o was, the "smaller" the flow index ϕ would be.

In this work, we are to prove this assertion for families of graphs embedded in any given surface.

Theorem 1.1. *Let S be any given surface and ϵ be a positive real number. Then there is a function $f_S(\epsilon)$ (depends on the surface S and $\lim_{\epsilon \to 0} f_S(\epsilon) = \infty$) such that any graph G embedded in S with the odd-edge connectivity at least $f_S(\epsilon)$ admits a nowhere-zero circular $(2+\epsilon)$-flow.*

This theorem can also be considered as a partial result of the following conjectures.

Conjecture 1.1. (Jaeger [8], also see [9], or Conjecture 9.1.5 in [19]). *Let t be a positive integer. Every graph with edge connectivity at least $4t+1$ admits a nowhere-zero circular $(2+\frac{1}{t})$-flow.*

Conjecture 1.2. (Goddyn [7], Seymour [13]). *For any given $\epsilon > 0$, every graph with a sufficiently large edge connectivity admits a nowhere-zero circular $(2+\epsilon)$-flow.*

Actually, Theorem 1.1 is a corollary of the following more detailed result (Theorem 1.2).

Theorem 1.2. *Let S be a surface with the Euler characteristic c_S and k be a positive integer and G be a graph embedded in S with the odd-edge connectivity λ_o.*

(1) *When $c_S > 0$, if*

$$\lambda_o \geq 5(2k-1)+1 = 10k-4,$$

then G admits a nowhere-zero circular $\frac{2k+1}{k}$-flow;

(2) *When $c_S \leq 0$, if*

$$\lambda_o \geq (2k-1)\frac{5+\sqrt{49-24c_S}}{2}+1,$$

then G admits a nowhere-zero circular $\frac{2k+1}{k}$-flow.

2. VERTEX SPLITTING THAT MAINTAINS EMBEDDING AND ODD EDGE CONNECTIVITY

We are to present a key lemma of the work about a vertex-splitting operation that maintains the odd-edge connectivity and the embedding. This lemma makes the reduction method possible in the proof of the main theorem, and is expected to have much more applications in many related studies.
Notations: Let $U \subset V(G)$. The set of all edges with exactly one endvertex in U is denoted by $\delta(U)$. Let $X, Y \subset V(G)$ and $X \cap Y = \emptyset$. The set of all edges with one endvertex in X and another endvertex in Y is denoted by $(X \sim Y)$. (Obviously, $\delta(X) = (X \sim (V(G)\backslash X))$.)

Note that vertex-splitting method has been used extensively in the study of various subjects of graph theory. The following is a well-known and widely used lemma due to Fleischner ([4, 5, 6]) that maintains bridgeless (odd-edge connectivity at least 3) and the embedding.

Lemma 2.1. (Fleischner, [4,5,6]). *Let $G = (V, E)$ be a bridgeless graph. Assume that there is a vertex $v_a \in V(G)$ such that $d(v_a) \geq 4$. Arbitrarily label the edges of G incident with v_a as $\{e_1, \ldots, e_b\}$ (where $b = d(v_a)$). Then there is an integer $i \in \{1, \ldots, b\}$ such that the new graph obtained from G by splitting e_i and e_{i+1}* (mod *b*) *away from v_a remains bridgeless.*

Other related splitting lemmas can also be found in [12], etc. The following result is a key lemma (Lemma 2.2) of the paper, which generalizes the vertex splitting lemma of Fleischner (Lemma 2.1).

Lemma 2.2. *Let $G = (V, E)$ be a graph with odd edge connectivity λ_o. Assume that there is a vertex $v_a \in V(G)$ such that $d(v_a) \neq \lambda_o$ and $\neq 2$. Arbitrarily label the edges of G incident with v_a as $\{e_1, \ldots, e_b\}$ (where $b = d(v_a)$). Then there is an integer $i \in \{1, \ldots, b\}$ such that the new graph obtained from G by splitting e_i and e_{i+1}* (mod *b*) *away from v_a remains of odd edge connectivity λ_o.*

Proof. Assume v_a is the vertex described in the theorem that no pair of such edges can be split away from v_a. That is, for each $i \in \{1, \ldots, b\}$, the new graph G_i obtained from G by splitting e_i and e_{i+1} away from v_a has an odd edge cut of size $\lambda_o - 2$. Thus, for each i, G has an edge cut $T_i = (X_i \sim Y_i)$ separating G into two parts $\{X_i, Y_i\}$ such that $|T_i| = \lambda_o$ and $v_a \in X_i$ and $e_i, e_{i+1} \in \delta(X_i)$. Now we define *critical edge cut around v_a* as follows. An edge cut $T = (X \sim Y)$ of G, which separates G into $\{X, Y\}$, is called *a critical edge cut of G around v_a* if

(i) $|T| = \lambda_o$;
(ii) $v_a \in X$;
(iii) $|\delta(v_a) \cap \delta(X)| > 0$;
(iv) $|\delta(v_a) \cap E(X)| > 0$.

The edge cut $T_i = (X_i \sim Y_i)$ we had before (containing e_i and e_{i+1}) is a critical edge cut of G around v_a.

I. Choose a critical edge cut $T^* = (X^* \sim Y^*)$, which separates G into two parts $\{X^*, Y^*\}$ such that $|\delta(X^*) \cap \delta(v_a)|$ is *as large as possible* among all critical edge cuts around the vertex v_a.

II. Note that $d(v_a) > 2$. Let $q \in \{1, \ldots, b\}$ such that $e_q \in E(X^*)$ and $e_{q+1} \in \delta(X^*)$. Let $T_q = (X_q \sim Y_q)$ be a critical edge cut of G around v_a which separates G into two parts $\{X_q, Y_q\}$. (The edge-cut T_q is defined in the first paragraph of this proof: splitting $\{e_q, e_{q+1}\}$ away from v_a, the resulting graph has an odd edge cut $T_q \setminus \{e_q, e_{q+1}\}$ of size smaller than λ_o.) Here $|T_q| = \lambda_o$.

III. Let $A = X^* \cap X_q$, $B = Y^* \cap X_q$, $C = X^* \cap Y_q$, and $D = Y^* \cap Y_q$. Obviously,

$$v_a \in A,$$
$$e_q \in (A \sim C),$$

and

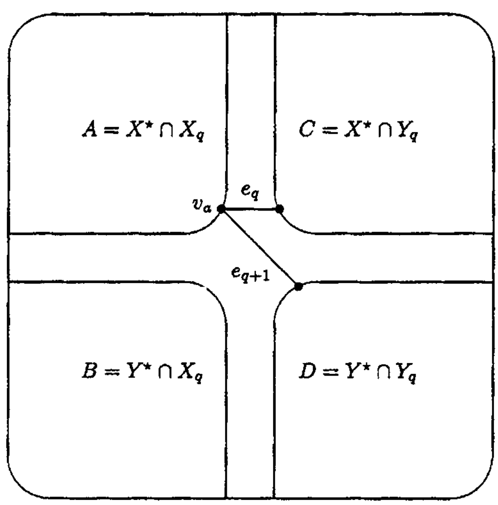
$$e_{q+1} \in (A \sim D). \tag{1}$$

Since both $\delta(X^*)$ and $\delta(X_q)$ are critical edge cuts around v_a, we have

$$\begin{aligned} &|(A \sim B)| + |(A \sim D)| + |(B \sim C)| + |(C \sim D)| \\ &= |\delta(A \cup C)| = |\delta(X^*)| = \lambda_o \equiv 1, (\text{mod}\, 2) \end{aligned} \tag{2}$$

and

$$\begin{aligned} &|(A \sim C)| + |(A \sim D)| + |(B \sim C)| + |(B \sim D)| \\ &= |\delta(A \cup B)| = |\delta(X_q)| = \lambda_o \equiv 1, (\text{mod } 2). \end{aligned} \tag{3}$$



IV. Case 1. $|\delta A|$ is *odd*. That is, in this case,

$$|\delta(A)| = |(A \sim B)| + |(A \sim C)| + |(A \sim D)| \equiv 1 (\text{mod } 2). \tag{4}$$

The binary sum of (4) and (2) yields that

$$|(A \sim C)| + |(B \sim C)| + |(C \sim D)| \equiv 0 (\text{mod } 2). \tag{5}$$

Therefore,

$$|\delta(C)| \equiv 0 (\text{mod } 2)$$

since $\delta(C) = (A \sim C) \cup (B \sim C) \cup (C \sim D)$.

The binary sum of (4) and (3) yields that

$$|(A \sim B)| + |(B \sim C)| + |(B \sim D)| \equiv 0 (\text{mod } 2).$$

Therefore,

$$|\delta(B)| \equiv 0 (\text{mod } 2)$$

since $\delta(B) = (A \sim B) \cup (B \sim C) \cup (B \sim D)$.

The binary sum of (5) and (3) yields that

$$|(A \sim D)| + |(B \sim D)| + |(C \sim D)| \equiv 1 (\text{mod } 2).$$

Therefore,

$$|\delta(D)| \equiv 1 (\text{mod } 2) \tag{6}$$

since $\delta(D) = (A \sim D) \cup (B \sim D) \cup (C \sim D)$.

Since the odd edge connectivity of G is λ_o and $|\delta(A)|$, $|\delta(D)|$ are odd (by (6)), we have that

$$|(A \sim B)| + |(A \sim C)| + |(A \sim D)| = |\delta(A)| \geq \lambda_o. \tag{7}$$

and

$$|(A \sim D)| + |(B \sim D)| + |(C \sim D)| = |\delta(D)| \geq \lambda_o. \tag{8}$$

Combining (7) and (2), we have that

$$|(A \sim C)| \geq |(C \sim D)| + |(B \sim C)|. \tag{9}$$

Similarly, combining (7) and (3), we have that

$$|(A \sim B)| \geq |(B \sim D)| + |(B \sim C)|, \tag{10}$$

and combining (8) and (3), we have that

$$|(C \sim D)| \geq |(A \sim C)| + |(B \sim C)|, \tag{11}$$

and combining (8) and (2), we have that

$$|(B \sim D)| \geq |(A \sim B)| + |(B \sim C)|. \tag{12}$$

The sum of (9), (10), (11), and (12) yields that

$$\begin{aligned} &|(A \sim B)| + |(A \sim C)| + |(B \sim D)| + |(C \sim D)| \\ &\geq |(A \sim B)| + |(A \sim C)| + |(B \sim D)| + |(C \sim D)| + 4|(B \sim C)|. \end{aligned} \tag{13}$$

Hence,

$$|(B \sim C)| = 0$$

and moreover, all equalities $(7), \ldots, (13)$ hold. That is, (7) and (8) became

$$|\delta(A)| = |\delta(D)| = \lambda_o. \tag{14}$$

Note that $|\delta(A) \cap \delta(v_a)|$ is larger than $|\delta(X^*) \cap \delta(v_a)|$, since $\delta(A) \cap \delta(v_a) \supseteq [\delta(X^*) \cap \delta(v_a)] \cup \{e_q\}$ (by (1)). If $\delta(A) = (A \sim [B \cup C \cup D])$ is a critical cut, then it would contradict the choice of T^*. Therefore, $\delta(A)$ is not critical. By (14), the cut $\delta(A)$ satisfies (i), (ii), and (iii) of the definition for critical cut. So, condition (iv) for critical cut must fail. That is,

$$\delta(v_a) \cap E(A) = \emptyset.$$

Then $d(v_a) < \lambda_o$, since $|\delta(A)| = \lambda_o$ (by (14)) and $d(v_a) \neq \lambda_o$. Moreover, $A \setminus \{v_a\} \neq \emptyset$. Since the odd edge connectivity of G is λ_o, the degree $d(v_a)$, which is smaller than λ_o, must be even. So, $\delta(A \setminus \{v_a\}) = \delta(A) \setminus \delta(v_a)$ is an odd edge cut, since $\delta(v_a) \subset \delta(A)$ and $|\delta(A)| = \lambda_o$ is odd and $|\delta(v_a)| < \lambda_o$ is even. But $|\delta(A \setminus \{v_a\})| < \lambda_o$. This contradicts that the odd edge connectivity of G is λ_o.

V. Case 2. $|\delta A|$ *is even.* That is, in this case,

$$|\delta(A)| = |(A \sim B)| + |A \sim C)| + |(A \sim D)| \equiv 0 \,(\mathrm{mod}\ 2). \tag{15}$$

The binary sum of (15) and (2) yields that

$$|(A \sim C)| + |(B \sim C)| + |(C \sim D)| \equiv 1 (\mathrm{mod}\ 2). \tag{16}$$

Therefore,

$$|\delta(C)| \equiv 1(\text{mod } 2) \tag{17}$$

since $\delta(C) = (A \sim C) + (B \sim C) + (C \sim D)$.

The binary sum of (15) and (3) yields that

$$|(A \sim B)| + |(B \sim C)| + |(B \sim D)| \equiv 1(\text{mod } 2).$$

Therefore,

$$|\delta(B)| \equiv 1(\text{mod } 2) \tag{18}$$

since $\delta(B) = (A \sim B) + (B \sim C) + (B \sim D)$.

The binary sum of (16) and (3) yields that

$$|(A \sim D)| + |(B \sim D)| + |(C \sim D)| \equiv 0(\text{mod } 2).$$

Therefore,

$$|\delta(D)| \equiv 0(\text{mod } 2)$$

since $\delta(D) = (A \sim D) + (B \sim D) + (C \sim D)$.

Since the odd edge connectivity of G is λ_o and $|\delta(B)|$ and $|\delta(C)|$ are odd (by (17), (18)),

$$|(A \sim C)| + |(B \sim C)| + |(C \sim D)| = |\delta(C)| \geq \lambda_o. \tag{19}$$

$$|(A \sim B)| + |(B \sim C)| + |(B \sim D)| = |\delta(B)| \geq \lambda_o. \tag{20}$$

Combining (19) and (2), we have that

$$|(A \sim C)| \geq |(A \sim B)| + |(A \sim D)|; \tag{21}$$

Combining (20) and (3) we have that

$$|(A \sim B)| \geq |(A \sim C)| + |(A \sim D)|; \tag{22}$$

Combining (19) and (3), we have that

$$|(C \sim D)| \geq |(B \sim D)| + |(A \sim D)|; \tag{23}$$

Combining (20) and (2), we have that

$$|(B \sim D)| \geq |(C \sim D)| + |(A \sim D)|. \tag{24}$$

The sum of (21), (22), (23), and (24) yields that

$$|(A \sim B)| + |(A \sim C)| + |(B \sim D)| + |(C \sim D)|$$
$$\geq |(A \sim B)| + |(A \sim C)| + |(B \sim D)| + |(C \sim D)| + 4|(A \sim D)|.$$

Hence,

$$|(A \sim D)| = 0.$$

This contradicts that $(A \sim D) \neq \emptyset$, since $e_{q+1} \in (A \sim D)$ (by (1)). ■

Remarks about embedding. We notice that the edges of $\delta(v_a)$ are labeled in an arbitrary order in Lemma 2.2, and the pair of edges split away from v_a have consecutive subscripts. Thus, for an embedded graph, edges of $\delta(v_a)$ can be locally ordered clockwisely around one side of the surface. This implies that the splitting operation described in Lemma 2.2 maintains the embedding in the same surface.

3. PROOF OF A MAIN THEOREM

Definition 3.1. *Let G be a graph embedded in a surface S. The embedding is called a 2-cell embedding if every face of G in S is homeomorphic to the open disk.*

It is well-known fact that every connected graph has a 2-cell embedding in some surface.

Euler Formula. *For a connected graph G embedded in a surface S,*

$$|V(G)| + |F(G)| \geq |E(G)| + c_S$$

where $V(G)$, $E(G)$, and $F(G)$ are the sets of vertices, edges and faces of G respectively, c_S is the Euler characteristic of surface S. Moreover, the equality holds if and only if the embedding is a 2-cell embedding.

It is also well known that $c_S = 2$ if S is the sphere, $c_S = 1$ if S is the projective plane, and $c_S = 0$ if S is a torus.

We need the following lemmas in the proof of the main theorem.

Lemma 3.1. *Let G be a simple graph embedded in a surface with the Euler characteristic c_S. Then*

$$\delta(G) \leq 5 \tag{1}$$

when $c_S > 0$;

$$\delta(G) \leq \frac{5 + \sqrt{49 - 24c_S}}{2} \tag{2}$$

when $c_S \leq 0$.

Proof. The first part is a well-known fact that can be concluded directly from the Euler formula. The following is the proof of the second part which is also based on the Euler formula.

Let V, E, and F be the sets of vertices, edges and faces of a connected graph G embedded in the surface S. Since G is simple,

$$|F(G)| \leq \frac{2|E(G)|}{3}.$$

Thus, by the Euler formula that

$$|V(G)| + |F(G)| \geq |E(G)| + c_S,$$

we have that

$$|V(G)| + \frac{2|E(G)|}{3} \geq |E(G)| + c_S$$

$$|V(G)| \geq \frac{|E(G)|}{3} + c_S$$

$$|V(G)| + (-c_S) \geq \frac{|E(G)|}{3}$$

$$6(1 + \frac{(-c_S)}{|V(G)|}) \geq \frac{2|E(G)|}{|V(G)|}.$$

Let $d_{\text{ave}}(G) = \frac{2|E(G)|}{|V(G)|}$ (the average degree of G). That is,

$$6\left(1 + \frac{-c_S}{d_{\text{ave}}(G) + 1}\right) \geq d_{\text{ave}}(G)$$

since c_S is non-positive and G is simple and $|V(G)| \geq d_{\text{ave}}(G) + 1$. Hence,

$$(d_{\text{ave}}(G))^2 - 5d_{\text{ave}}(G) + (6c_S - 6) \leq 0.$$

Solve the inequality, we have that

$$d_{\text{ave}}(G) \leq \frac{5 + \sqrt{49 - 24c_S}}{2}.$$ ■

Definition 3.2. *Let $G = (V, E)$ be a graph. An ordered pair (D, f) is called a modular k-flow if D is an orientation of $E(G)$ and $f : E(G) \mapsto Z_k$ (where Z_k is the cyclic group of order k) such that the difference of the total in-flow and the total out-flow at every vertex is the zero of the additive group Z_k.*

Theorem 3.1. (Tutte [15], or see [19] Theorem 1.3.4). *Let D be an orientation of a graph G. If the graph G admits a nowhere-zero modular h-flow (D,f'), then the graph G admits a nowhere-zero integer h-flow (D,f'') such that $f'(e) \equiv f''(e) (\mathrm{mod}\ h)$.*

Notation. Let G be a graph and $x, y \in V(G)$. The *multiplicity* between x and y, denoted by $m(xy)$, is the number of edges (parallel edges) joining x and y.

Lemma 3.2. *Let k be an integer and let G be a graph and $v_1, v_2 \in V(G)$. Assume that $m(v_1v_2) \geq 2k$. Let H be the graph obtained from G by contracting precise $2k$ parallel edges between v_1 and v_2. If H admits a nowhere-zero circular $\frac{2k+1}{k}$-flow, then G admits a nowhere-zero circular $\frac{2k+1}{k}$-flow as well.*

Proof. Let $\{e_1, \ldots, e_{2k}\}$ be a subset of edges joining v_1 and v_2, and let H be the graph obtained from G by contracting all edges of $\{e_1, \ldots, e_{2k}\}$. Let (D,f) be a nowhere-zero circular $(\frac{2k+1}{k})$-flow of H with $f : E(G) \mapsto \{\pm k, \pm(k+1)\}$.

Note that (D,f) is a "flow" of G with at most two possibly "unbalanced" vertices v_1 and v_2 and zero-weight at every unoriented edge of $\{e_1, \ldots, e_{2k}\}$. Let

$$d \equiv \sum_{e \in E^+(v_1)} f(e) - \sum_{e \in E^-(v_1)} f(e) = \sum_{e \in E^-(v_2)} f(e) - \sum_{e \in E^+(v_2)} f(e) \ \mathrm{mod}(2k+1).$$

Without loss of generality, assume that

$$0 \leq |d| \leq k.$$

Then extend the orientation D and the weight f to the edges $\{e_1, \ldots, e_{2k}\}$ as follows:

Each e_i is oriented from v_2 to v_1 for $i \leq k - d$, and each e_j is oriented from v_1 to v_2 for each $i > k - d$; $f(e_i) = k$ for each $i \in \{1, \ldots, 2k\}$.

Under the extended orientation D and weight f,

$$\sum_{e \in E^+(v_1)} f(e) - \sum_{e \in E^-(v_1)} f(e)$$

$$\equiv d - k(k-d) + k[2k - (k-d)](\mathrm{mod}(2k+1))$$

$$= d + d(2k)$$

$$\equiv d + d(-1) \equiv 0 \ \mathrm{mod}(2k+1).$$

So, the resulting orientation and weight yield a nowhere-zero modular $(2k+1)$-flow (D,f) with $f(e) \in \{\pm k, \pm(k+1)\}$ for every edge e. By Theorem 3.1, G admits a nowhere-zero $(2k+1)$-flow (D,f') with $f'(e) \equiv f(e)$ for every edge e. Obviously (D,f) is a nowhere-zero circular $\frac{2k+1}{k}$-flow. ■

Notation. Let G be a graph. The *underlying graph* of G, denoted by $\bar{G}$, is obtained from G by replacing every maximal induced path with an edge. The *background graph* of G, denoted by $\tilde{G}$, is the simple graph obtained from G by replacing every parallel edges of G with a single edge.

Proof of Theorem 1.2. For a given surface S with the Euler characteristic c_S, let G be a connected graph embedded in S and a counterexample to the theorem with the least number of edges.

If there is a vertex $v_1 \in V(G)$ with $d(v_1) \neq \lambda_o$, then by Lemma 2.2, two edges of $E(G)$ can be split away from v_1 such that the resulting graph G' is still embedded in S and the odd-edge connectivity of G' is still at least λ_o. Since G is a smallest counterexample, the underlying graph $\bar{G}'$ of G', which has less edges than G, must admit a nowhere-zero circular $\frac{2k+1}{k}$-flow. Therefore, G admits a nowhere-zero circular $(\frac{2k+1}{k})$-flow as well. This contradicts the assumption that G is a counterexample. So, G is a λ_o-regular graph.

Let $\tilde{G}$ be the background graph of G and let v_2 be a vertex of $\tilde{G}$ with the minimum degree $\delta(\tilde{G})$. By Lemma 3.1, $\delta(\tilde{G}) \leq \frac{5+\sqrt{49-24c}}{2}$ when $c_S \leq 0$ or $\delta(\tilde{G}) \leq 5$ when $c_S > 0$.

Note that

$$\lambda_o \geq (2k-1)\delta(\tilde{G}) + 1.$$

We have that there are parallel edges joining v_2 and another vertex v_3 with the multiplicity $m(v_2v_3)$ at least $2k$. Let H be the graph obtained from G by contracting $2k$ of those parallel edges. Since the odd-edge connectivity of H remains as λ_o and the graph H remains in the surface S, the graph H admits a nowhere-zero circular $\frac{2k+1}{k}$-flow. Thus G admits a nowhere-zero circular $\frac{2k+1}{k}$-flow by Lemma 3.2. This is a contradiction again and completes the proof. ■

4. $(2+\epsilon)$-FLOWS AND 5-FLOW

Let us define a sequence $\{\theta_i\}_{i=1}^{\infty} = \{\frac{1}{3}, 2, 7, \ldots\}$ as follows: $\theta_1 = \frac{1}{3}$ and $\theta_{i+1} = 3\theta_i + 1$ for each natural number *i*.

Statement *S(i). Every graph G with odd edge connectivity* $\lambda_o \geq 3^i$ *admits a nowhere-zero circular* $(2+\frac{1}{\theta_i})$*-flow.*

Note that each statement $S(i)$ is a weak version of Conjecture 1.6 and the statement $S(1)$ is the Tutte's 5-flow conjecture ([16]). The following proposition indicates that the verification of any statement $S(i)$ implies the 5-flow conjecture.

Proposition 4.1. *The statement* $S(i+1)$ *implies the statement* $S(i)$, *for every integer* $i = 1, 2, \ldots$.

Proof. Let G be a graph with the odd edge connectivity $\lambda_o \geq 3^i$ and D be an arbitrary orientation of G. Let $3G$ be the graph obtained from G by replacing each edge e of G with three parallel edges e_1, e_2, e_3. Without any confusion, let D be the orientation of $3G$ that the three new parallel edges e_1, e_2, e_3 have the same orientation as that of e in G.

Obviously, the odd edge connectivity of $3G$ is at least 3^{i+1}. With the assumption of the statement $S(i+1)$, the graph $3G$ admits a nowhere-zero circular $(\frac{2\theta_{i+1}+1}{\theta_{i+1}})$-flow (D, f') where

$$f' : E(3G) \mapsto \{\pm\theta_{i+1}, \pm(\theta_{i+1}+1)\}.$$

Define f as a weight of $E(G)$ follows:

$$f(e) = \sum_{\mu=1}^{3} f'(e_\mu) \bmod (2\theta_{i+1}+1).$$

Note that

$$\theta_{i+1} + 1 \equiv -\theta_{i+1} \bmod (2\theta_{i+1}+1).$$

It is easy to see that

$$f(e) \in \{\pm\theta_{i+1}, \pm 3\theta_{i+1}\} \equiv \{\pm\theta_{i+1}, \pm(\theta_{i+1}-1)\} \bmod (2\theta_{\mathrm{i+1}}+1)$$

for every edge $e \in E(G)$. That is, by Theorem 3.4, G admits an integer $(2\theta_{i+1}+1)$-flow (D'', f'') with

$$f'' : E(G) \mapsto \{\pm(\theta_{i+1}-1), \pm\theta_{i+1}, \pm(\theta_{i+1}+1), \pm(\theta_{i+1}+2)\}.$$

So, (D'', f'') is a nowhere-zero circular $\frac{2\theta_{i+1}+1}{\theta_{i+1}-1}$-flow of G.

Since $\theta_{i+1} = 3\theta_i + 1$, we have that

$$\frac{2\theta_{i+1}+1}{\theta_{i+1}-1} = \frac{2\theta_i+1}{\theta_i}.$$

That is, G admits a nowhere-zero circular $(\frac{2\theta_i+1}{\theta_i})$-flow. ■

5. REMARKS

The dual version of Theorem 1.2 for planar graph is the result that proved in [11].

Theorem 5.1 (Klostermeyer and Zhang [11]). *Let G be a planar graph. If the odd girth of G is at least* $10k-3$, *then G is* $\frac{2k+1}{k}$*-colorable.*

Note that Youngs ([18]) and Klavžar, Mohar ([10]) constructed families of 4-chromatic graphs in the projective plane and torus with an arbitrary large odd girth. Thus, Theorem 5.1 cannot be generalized for non-planar graphs. However, the circular flow problem, as the "dual" of the circular coloring problem, is generalized to graphs embedded in any surface (Theorem 1.2). Notice that the embedding requirement plays a key role in the proof that theorem. Could we prove the theorem without any requirement of embedding?

Conjecture 5.1. *Let ϵ be a positive real number. Then there is a function $f(\epsilon)$* ($\lim_{\epsilon\to 0} f(\epsilon)=\infty$) *such that any graph G with the odd-edge connectivity at least $f(\epsilon)$ admits a nowhere-zero circular $(2+\epsilon)$-flow.*

Conjecture 5.1 is a revised version of Conjecture 1.2.

The following conjecture is a revised version of a conjecture by Jaeger (Conjecture 1.1).

Conjecture 5.2. *Let t be a positive integer. Every graph with odd edge connectivity at least $4t+1$ admits a nowhere-zero circular $(2+\frac{1}{t})$-flow.*

When $t=1$, Conjecture 1.1 (as well as Conjecture 5.2) is the 3-flow conjecture by Tutte ([1, 14]); and when $\epsilon=1$, Conjecture 1.2 is the weak 3-flow conjecture by Jaeger ([9], or see [19] Conjecture 4.3.1).

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